DAVID ESSNER FINALS 2002-2003

The point values for the problems are: (1) 15; (2) 20; (3) 15; (4) 25; (5) 25.
The use of a calculator is permitted only on problems 1(c) and 2. Graphic features of a calculator are *not* permitted.

I Integers Which Are Both Squares and Cubes Problem

For the purpose of this problem let a positive integer be called *dual* if it greater than 1 and is the square of a positive integer and also the cube of a positive integer.

(a) Find two dual positive integers.
(b) Find with proof a (necessary and sufficient) condition for an integer to be a dual integer.
(c) **(Calculator Problem)** Find all dual integers less than 600,000.
(d) Do you think the sum of two dual integers could be a dual integer? Why?

II The Modulo Problem (Calculators *not* permitted)

In this problem if n is a positive integer then $n \; mod \; 9$ denotes the remainder of the division of n by 9.

(a) Let m,n be positive integers; prove that if $n \; mod \; 9 = m \; mod \; 9$ then $(n+1) \; mod \; 9 = (m+1) \; mod \; 9$.

(b) Let m,n be positive integers. Prove that if $2^n \; mod \; 9 = 2^{n+m} \; mod \; 9$ then $2^{n+1} \; mod \; 9 = 2^{n+m+1} \; mod \; 9$

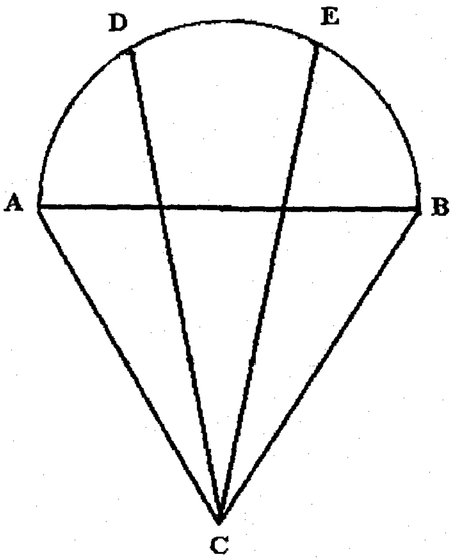
(c) What are the possible remainders if n is a positive integer and 2^n is divided by 9? Justify your answer.

(d) **Using (b)** find the value of $2^{420,000,003} \; mod \; 9$; explain and justify your method of solution. (Calculators **NOT** permitted)

III The Ice Cream Cone Problem (A calculator problem)

In the figure ABC is an equilateral triangle with sides of length 2 and AB is a diameter of the pictured semicircle. If the arcs AD, DE and EB are equal in length, find

(a) the measure of angle $\angle DCE$ to the nearest 0.1 degree.
(b) the length of CE to the nearest .001



IV The Basketball Ordered Point Sequence Problem

(a) In 1950 a basketball player could score in two ways –
(i) 2 points for a field goal (ii) 1 point for a free throw

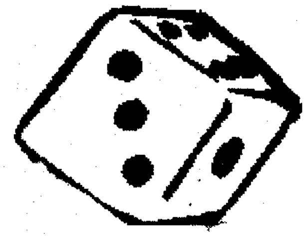
For $n = 1,2,3,\ldots$ let S_n denote the number of ordered ways a player could score n points. For example $S_4 = 5$ since there are the orderings: (1,1,1,1), (1,1,2), (1,2,1), (2,1,1). (2,2).

(a1) Find S_3 and S_5.
(a2) For $n > 5$ find a method for determining S_n from (not necessarily all of) S_1, S_2, … , S_{n-1}. Explain why your method works.
(a3) Find S_{11} using your method in (a2). Do not try to list the sequences as there are more than 100.

(b) In the present day a basketball player can score points in three ways:
(i) 3 points for a long field goal (ii) 2 points for a regular field goal
(iii) 1 point for a free throw.

Let T_n be the number of ordered ways a player could score n points. For example $T_3 = 4$ since there are the four orderings (1,1,1), (2,1), (1,2), (3). Find T_{11} and explain your method of solution.

V. The Fair Game Gambling Problem



(a) Players A and B alternately roll a 6 sided die, A going first. The first player to roll a 6 is the winner.
   (a1) Find the probability that A wins after A makes 2 or fewer rolls
   (a2) Find the probability A is the winner

(b) Suppose in (a) that A goes first and gets one roll, then B gets two rolls, then A gets one roll, then B gets two rolls continuing so that each time A gets one roll followed by B getting two rolls. What is the probability that A wins?

A gambling game among two or more players is a *fair game* if each player has equal probability of winning the game.

(c) Suppose A and B play with a die which is 'loaded' such that the probability of rolling a 6 is p. If A goes first and gets one try, then B gets two tries, then A gets one try, then B gets two tries, continuing until there is a winner; find the value of p so that this is a fair game.

(d) Suppose A and B play with a die which is loaded such that the probability of rolling a 6 is α and the probability of rolling a 5 is β, where $0 < \alpha < \beta < 1$. A and B alternately roll the loaded die, A going first, and each having one roll at their turn. If A rolls a 6 before B rolls a 5 then A is the winner; otherwise B is the winner.
   (d1) Find the relationship between α and β in order that this is a fair game.
   (d2) Determine the values of α such that there is a value of β to make this a fair game.