Precalculus Individual Test

February 18,2006 Middleton Invitational

For all questions, choice **E** is **NOTA**, meaning "None Of These Answers."

<u>NO CALCULATOR</u>

All inverse trigonometric functions are one-to-one, with the standard textbook domains and ranges.

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1. Evaluate: $\log_4 1$

A) 0 B) 0.25 C) 1 D) 4 E) NOTA

2. Let a be a constant such that $0 < a < 1$. The graph of which of the following yields a vertical compression of the graph of $y = f(x)$?

A) $y = a \cdot f(x)$ B) $y = \dfrac{f(x)}{a}$ C) $y = f(a \cdot x)$

D) $y = f\left(\dfrac{x}{a}\right)$ E) NOTA

3. How many of the following four functions are continuous at $x = -3$?

$$f(x) = 2$$

$$g(x) = \frac{3x+9}{2x^2+7x+3}$$

$$h(x) = \begin{cases} 2x+1, & \text{if } x \le -3 \\ -x^2+4, & \text{if } x > -3 \end{cases}$$

$$j(x) = \sqrt{x-3}$$

A) 1 B) 2 C) 3 D) 4 E) NOTA

4. Evaluate: $\text{Arccos}\left(\cos\left(\dfrac{4\pi}{3}\right)\right)$

A) $-\dfrac{\pi}{3}$ B) $\dfrac{\pi}{3}$ C) $\dfrac{2\pi}{3}$ D) $\dfrac{4\pi}{3}$ E) NOTA

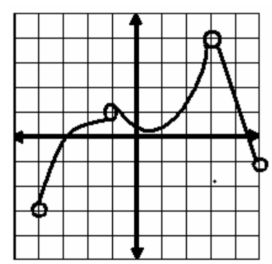
5. The horizontal asymptote of $f(x) = \dfrac{3x^4+8}{2x^5+6}$ has the equation $y = P$. Find P.

A) $-\dfrac{3}{2}$ B) 0 C) $\dfrac{3}{2}$ D) 2 E) NOTA

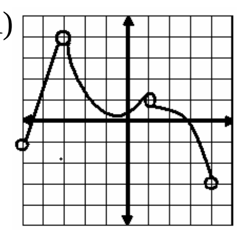
6. Simplify $\dfrac{\left(x^2\right)^y}{x^2}$, where $xy \neq 0$.

A) 1 B) y C) $x^{2y} - x^2$ D) x^y E) NOTA

7. The graph of $p(x)$ is shown to the right. Which is the graph of $p(|x|)$?



A)



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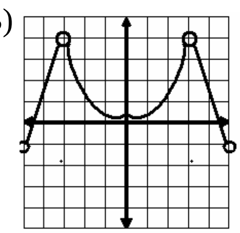

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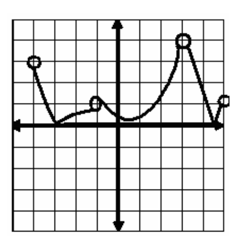
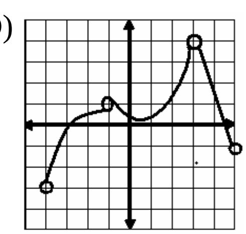


D)



E) NOTA

8. Let A and B be the measures of the acute angles of a right triangle. If $\csc(A) = \sqrt{5}$, then evaluate $5\csc(A) \bullet |\cos(2A + B)|$.

A) 5 B) 7 C) 9 D) 11 E) NOTA

9. Let $M(x) = (x-3)^4 \left(3x + 0.\overline{3}\right)^5 \left(\frac{x}{3} + 1\right)^3$.

Now, let **d** be the degree of M, let **b** be the y-intercept of M, and let **c** be the leading coefficient of M. Find the product **bcd .**

A) $\frac{4}{3}$ B) 4 C) 12 D) 36 E) NOTA

10. The graph of $h(x)$ is tangent to the x-axis at the point (3, 0). Which of the following could be the equation of $h(x)$?

A) $h(x) = (x-3)^6$ B) $h(x) = (x-3)^7$
C) $h(x) = (x+3)^6$ D) $h(x) = (x+3)^7$
E) NOTA

11. Let $f(x) = \sec x$, $g(x) = \tan^{-1} x$, and $h(x) = (f \cdot g)(x)$. Which of the following statements are true ?

I. $f(x)$ is an odd function.
II. $g(x)$ is an odd function.
III. $h(x)$ is an odd function.

A) I & II only B) II & III only C) I & III only D) I, II, & III E) NOTA

12. Find $\log\left(\frac{M^2}{N}\right)$, if $\log M = 7$ and $N = 100$.

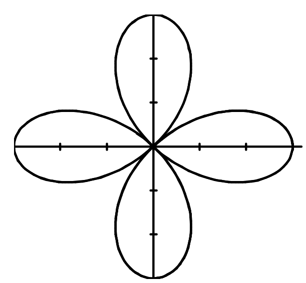
A) 7 B) 12 C) 24.5 D) 47 E) NOTA

13. How many values of θ on the interval $[0, 2\pi)$ satisfy the equation $\sin(3\theta - 0.25\pi) = -\frac{\sqrt{3}}{2}$?

A) 4 B) 5 C) 6 D) 7 E) NOTA

14. Which of the following polar equations could produce the graph shown below?

A) $r = 3\cos(2\theta)$
B) $r = 3\cos(4\theta)$
C) $r = 3\sin(2\theta)$
D) $r = 3\sin(4\theta)$
E) NOTA



15. Evaluate $(\tan\beta)$, if $\frac{2\beta}{\pi}$ lies on the interval $[2,3]$ and $|\cos\beta| = \frac{12}{13}$.

A) $-\frac{5}{12}$ B) $-\frac{12}{5}$ C) $\frac{5}{12}$ D) $\frac{12}{5}$ E) NOTA

16. Let $h(\alpha) = 4\cos\left(\frac{3\pi}{5}\alpha - 1\right) + 2$. How many of the following four statements are true?

- The amplitude of h is 4.
- The period of h is $3.\overline{3}$.
- The phase shift of h is 1.
- h is a periodic function.

A) 1 B) 2 C) 3 D) 4 E) NOTA

17. Emily and Pedro are planning a dinner party, to which they will invite exactly 4 people. Each guest will be either one of Pedro's five best friends (all of them Elvis impersonators), or one of Emily's four ninja assassin disciples.
The guest list must include at least one person from each of these two groups of distinct people. How many distinct guest lists are possible?

A) 102 B) 116 C) 120 D) 126 E) NOTA

18. When the polar point $\left(4\sqrt{2}, \frac{11\pi}{12}\right)$ is expressed using rectangular coordinates, its abscissa is $a + b\sqrt{c}$ in simplest radical form.
Evaluate the sum $a + 2b + c$.

A) -3 B) 1 C) 5 D) 9 E) NOTA

19. What is the domain of $h(\theta) = \cot\theta$?
(In the choices, k represents an integer.)

A) $(-\infty, \infty)$ B) $\{x \mid x \neq k\pi\}$ C) $\left\{x \mid x \neq \frac{\pi}{2} + k\pi\right\}$
D) $(-\infty, -1] \cup [1, \infty)$ E) NOTA

20. $\frac{3}{4} + \frac{7}{16} + \frac{17}{64} + \frac{43}{256} + \frac{113}{1024} + \ldots =$

A) $\sqrt{3}$ B) $\frac{7}{4}$ C) 2 D) e E) NOTA

21. Evaluate: $(3\vec{i} + 2\vec{j}) \bullet (5\vec{i} - 4\vec{j})$

A) $8\left(\vec{i} - \vec{j}\right)$ B) $8\vec{i} + 2\vec{j}$ C) $15\vec{i} - 8\vec{j}$
D) $17\left(\vec{i} - \vec{j}\right)$ E) NOTA

22. In triangle YUM, $m\angle Y = 100°$, $m\angle M = 50°$, and $MU = 14$. Find the area of the triangle, to the nearest integer. Use the approximations shown below, if needed.

A) 28
B) 31
C) 35
D) 39
E) NOTA

$\sin 50° \approx \frac{7}{9}$	$\sin 100° \approx \frac{49}{50}$
$\cos 50° \approx \frac{16}{25}$	$\cos 100° \approx -\frac{17}{100}$

23. Let $f(x)=x^2+5x$. Find the average rate of change in $f(x)$ over the interval $[1,3]$.

A) 4.5 B) 7.5 C) 9 D) 15 E) NOTA

24. Let $g(x)$ be a cubic polynomial with relatively prime integral coefficients. If $(3-i)$ and 4 are roots of *g*, then evaluate $|g(1)|$.

A) 9 B) 15 C) 18 D) 25 E) NOTA

25. A wheel with a $\frac{6}{\pi}$-foot radius rolls along a smooth road. The wheel's angular velocity is 40 revolutions per minute. Find the wheel's linear speed, in miles per hour. Round to the nearest integer.

A) 3 mph B) 5 mph C) 7 mph D) 9 mph E) NOTA

26. Find the constant term in the expansion of $\left(2x^{-3}+x^4\right)^7$.

A) 84 B) 128 C) 280 D) 560 E) NOTA

27. To eliminate the *xy* term of the equation $x^2+2xy+4y^2-5=0$, the coordinate plane's axes will be rotated through the acute angle θ.

Evaluate $\cot(2\theta)$.

A) $-\frac{3}{2}$ B) $-\frac{1}{2}$ C) $\frac{1}{2}$ D) $\frac{3}{2}$ E) NOTA

28. Decompose $\frac{x+2}{x^3-2x^2+x}$ into partial fractions, and find the sum of the resultant numerators.

A) 1 B) 2 C) 3 D) 4 E) NOTA

29. Find the distance from the point (3, 1) to the line $y=-(0.5x+2.5)$.

A) $\sqrt{5}$ B) $2\sqrt{5}$ C) $\sqrt{15}$ D) $\frac{5\sqrt{5}}{3}$ E) NOTA

30. The equation $x^3=\sqrt{-1}$ has the solutions $\operatorname{cis}(A^\circ)$, $\operatorname{cis}(B^\circ)$, and $\operatorname{cis}(C^\circ)$, where $0\le A<B<C<360$.

Evaluate $(A+B+C)$.

A) 360 B) 450 C) 495 D) 540 E) NOTA

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Precalculus Individual Test	**SOLUTIONS**

1. (A) $4^0 = 1 \rightarrow \log_4 1 = 0$

2. (A) Multiplying a function by a number between 0 and 1 yields a vertical compression.

3. (B) Remember that for $P(x)$ to be continuous at $x=-3$, $P(-3)$ and $\lim_{x\to -3} P(x)$ must each exist, and they must be equal. *f* satisfies the conditions, since $f(x) = \lim_{x\to a} f(x) = 2$ everywhere. $g(-3)$ does not exist. *h* is good, since $\lim_{x\to -3} h(x) = -5 = h(-3)$. $j(-3)$ does not exist.

4. (C) $f(x) = \text{Arccos}(x)$ has a restricted range of $[0,\pi]$. $\text{Arccos}\left(\cos\frac{4\pi}{3}\right) = \text{Arccos}\left(-\frac{1}{2}\right)$, and the only value on $[0,\pi]$ that works is $\frac{2\pi}{3}$.

5. (B) For all rational functions where the polynomial in the denominator has a higher degree than the polynomial in the numerator.

6. (E) $\frac{\left(x^2\right)^y}{x^2} = \frac{x^{2y}}{x^2} = x^{2y-2}$.

7. (B) To get the graph of $f\left(|x|\right)$, discard the left side of the graph of *f* and replace it with a reflection of the right side over the *y*-axis.

8. (A) Using right-triangle trig, $\cos A = \sin B = \frac{2}{\sqrt{5}}$, and $\cos B = \sin A = \frac{1}{\sqrt{5}}$. You can either pummel through the double application of identities, or use a little finesse.

$$\cos(2A+B) = \cos(2A)\cos(B) - \sin(2A)\sin(B) = (\cos^2 A - \sin^2 A)(\cos B) - \left(2\sin A\cos A\right)(\sin B) = \left(\frac{4}{5}-\frac{1}{5}\right)\left(\frac{1}{\sqrt{5}}\right) - 2\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{3}{5\sqrt{5}} - \frac{8}{5\sqrt{5}} = -\frac{1}{\sqrt{5}}.$$

Since $B = \frac{\pi}{2} - A$, $\cos(2A+B) = \cos\left(2A + \frac{\pi}{2} - A\right) = \cos\left(A + \frac{\pi}{2}\right) = \cos A\cos\frac{\pi}{2} - \sin A\sin\frac{\pi}{2} = -\sin A = -\frac{1}{\sqrt{5}}$.

Either way, $5\csc A\left|\cos(2A+B)\right| = 5\sqrt{5}\left(\frac{1}{\sqrt{5}}\right) = 5$.

9. (D) **d** = 4 + 5 + 3 = 12, **b** $= (-3)^4\left(\frac{1}{3}\right)^5(1)^3 = \frac{1}{3}$, and **c** $= (1)^4(3)^5\left(\frac{1}{3}\right)^3 = 9$.

10. (A) Select the function where the factor $(x-3)$, corresponding to a root of 3, has an even multiplicity.

11. (B) *f* is even, and *g* is odd. An even function multiplied by an odd function yields an odd function.

12. (B) Note that $\log N = 2$. Then: $\log\left(\frac{M^2}{N}\right) = \log\left(M^2\right) - \log(N) = 2\log M - \log N = 14 - 2 = 12$.

13. (C) Let $x = 3\theta - \frac{\pi}{4} = \frac{12\theta - \pi}{4}$. Substitute; $\sin x = -\frac{\sqrt{3}}{2}$ yields the solutions $x = \frac{4\pi}{3} + 2k\pi = (4+6k)\frac{\pi}{3}$ & $x = \frac{5\pi}{3} + 2k\pi = (5+6k)\frac{\pi}{3}$, where k is an integer. So

$$\frac{12\theta - \pi}{4} = \frac{(4+6k)\pi}{3} \rightarrow 36\theta - 3\pi = 16\pi + 24k\pi \rightarrow \theta = \frac{(19+24k)\pi}{36}, \quad \text{and}$$

$$\frac{12\theta - \pi}{4} = \frac{(5+6k)\pi}{3} \rightarrow 36\theta - 3\pi = 20\pi + 24k\pi \rightarrow \theta = \frac{(23+24k)\pi}{36}.$$

The only values of k that yield θ on $[0, 2\pi)$ in each solution are 0, 1, and 2. So there are 6 total solutions.

14. (A) For even values of n, $r = A\cos(n\theta)$ produces an evenly distributed $2n$-petal graph (each petal having length A) , with one petal on the positive x-axis.

15. (C) $|\cos\beta| = \frac{12}{13} \rightarrow |\tan\beta| = \frac{5}{12}$. As $2 \le \frac{2\beta}{\pi} \le 3 \rightarrow \pi \le \beta \le \frac{3\pi}{2}$, β lies in Quadrant III, and $\tan\beta > 0$.

16. (C) Using the standard form $f(x) = A\cos\left(\omega(x-P)\right) + B$, $h(\alpha) = 4\cos\left(\frac{3\pi}{5}\left(\alpha - \frac{5}{3\pi}\right)\right) + 2$. The amplitude is $A = 4$, the period is $\frac{2\pi}{\omega} = \frac{2\pi}{1} \bullet \frac{5}{3\pi} = \frac{10}{3} = 3.\overline{3}$, the phase shift is $P = \frac{5}{3\pi}$, and all sinusoids are periodic.

17. (C) There are ${}_9C_4 = \frac{9!}{4!5!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5!}{(4\cdot 3\cdot 2)(5!)} = 126$ total possibilities. But the ${}_5C_4 = \frac{5!}{4!1!} = 5$ combinations from only Pedro's group and ${}_4C_4 = 1$ combination using only Emily's group must be subtracted.

18. (A) If you don't know your $\frac{\pi}{12}$ intervals of the unit circle, then trudge through the angle sum identity to find that $\cos\left(\frac{11\pi}{12}\right) = \cos\left(\frac{9\pi + 2\pi}{12}\right) = \cos\left(\frac{3\pi}{4} + \frac{\pi}{6}\right) = \left(\cos\frac{3\pi}{4}\cdot\cos\frac{\pi}{6}\right) - \left(\sin\frac{3\pi}{4}\cdot\sin\frac{\pi}{6}\right) = \frac{-\sqrt{2}-\sqrt{6}}{4}$. The abscissa ($x$-coordinate) is $r\cos\theta = \left(4\sqrt{2}\right)\left(\frac{-\sqrt{2}-\sqrt{6}}{4}\right) = -2 - 2\sqrt{3}$.

19. (B) $\cot\theta = \frac{\cos\theta}{\sin\theta}$, and is undefined where $\sin\theta = 0$, which is along the x-axis, at all values of $k\pi$.

20. (C) An insightful severing is required. The series equals $\frac{1+2}{4} + \frac{3+4}{16} + \frac{9+8}{64} + \frac{27+16}{256} + \frac{81+32}{1024} + \ldots$, so separate them. $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256} + \ldots$ is an Infinite Geometric Series where $a_1 = \frac{1}{4}$ and $r = \frac{3}{4}$; its sum is $\frac{.25}{1-.75} = 1$. $\frac{2}{4} + \frac{4}{16} + \frac{8}{64} + \frac{16}{256} + \ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is also an IGS. $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$; its sum is $\frac{.5}{1-.5} = 1$. And then 1 + 1 = 2.

21. (E) The dot product of two vectors is a scalar, and all the choices are vectors. The actual value is $(3)(5)+(2)(-4)=7$.

22. (D) $m\angle U=30°$, and the area of the triangle is $\frac{1}{2}(MU)(YU)\sin 30°$. We just need to find *YU*. Law of Sines. $\frac{\sin 100°}{14}=\frac{\sin 50°}{x} \rightarrow \frac{49}{50}x=14\left(\frac{7}{9}\right) \rightarrow x=\frac{100}{9}$. The area is $\frac{1}{2}\left(\frac{14}{1}\right)\left(\frac{100}{9}\right)\left(\frac{1}{2}\right)=\frac{350}{9}\approx 39$.

23. (C) $\frac{f(3)-f(1)}{3-1}=\frac{24-6}{2}=9$.

24. (B) A factor is $(x-4)$. Find the other factor. $x=3-i \rightarrow x-3=-i \rightarrow (x-3)^2=-1 \rightarrow x^2-6x+10=0$, so $g(x)=\pm(x-4)(x^2-6x+10)$, and $|g(1)|=|(1-4)(1-6+10)|=15$.

25. (B) The wheel's circumference is 12, so the wheel travels 480 feet per minute. Do unit conversions, and $\frac{480\text{ft}}{\text{min}}\times\frac{1\text{mile}}{5280\text{ft}}\times\frac{60\text{min}}{\text{hour}}=\frac{60}{11}\text{mph}\approx 5\text{mph}$.

26. (D) Note that the expansion is $ax^{-21}+bx^{-14}+cx^{-7}+d+...+hx^{28}$, so the 4th term is the constant. By the binomial theorem, the 4th term is ${}_7C_{(4-1)}\left(2x^{-3}\right)^{7-(4-1)}\left(x^4\right)^{(4-1)}=\frac{7!}{4!3!}\left(16x^{-12}\right)(x^{12})=(35)(16)=560$.

27. (A) In general form, $\cot(2\theta)=\frac{A-C}{B}=\frac{1-4}{2}=-\frac{3}{2}$.

28. (C) The denominator factors as $x(x-1)^2$. Remember that repeated factors must be represented appropriately. The partial fraction decomposition is $\frac{x+2}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$. Multiplying by the original denominator, $x+2=A(x-1)^2+Bx(x-1)+Cx=(A+B)x^2+(-2A-B+C)x+(A)$. Some quick substitution solves the system $\begin{cases}A+B=0\\-2A-B+C=1\\A=2\end{cases}$ to reveal that $A=2$, $B=-2$, and $C=3$.

29. (B) The distance from the point (M,N) to the line $Ax+By=C$ is $\frac{|AM+BN-C|}{\sqrt{A^2+B^2}}=\frac{|3+2+5|}{\sqrt{1+4}}=\frac{10}{\sqrt{5}}=2\sqrt{5}$.

30. (B) $\sqrt{-1}=i=\text{cis}\,90°$. By DeMoivre's Theorem, if $x^n=\text{cis}\,\theta$ for whole values of *n*, then $x=\text{cis}\frac{\theta}{n}\pm k\cdot\frac{360°}{n}$, where *k* is an integer, and only the first *n* values of *k* are distinct.
$x^3=\text{cis}\,90° \rightarrow x=\text{cis}\,30°,\ \text{cis}\,150°,\ \text{cis}\,270°$.

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ANSWER KEY

1.	**A**	**16.**	**C**
2.	**A**	**17.**	**C**
3.	**B**	**18.**	**A**
4.	**C**	**19.**	**B**
5.	**B**	**20.**	**C**
6.	**E**	**21.**	**E**
7.	**B**	**22.**	**D**
8.	**A**	**23.**	**C**
9.	**D**	**24.**	**B**
10.	**A**	**25.**	**B**
11.	**B**	**26.**	**D**
12.	**B**	**27.**	**A**
13.	**C**	**28.**	**C**
14.	**A**	**29.**	**B**
15.	**C**	**30.**	**B**

Middleton Invitational	February 2006
Precalculus Team (no calculator)	**Question # 1**

Each of the following statements has a value, indicated by the number in brackets.
Find the sum of the values of the true statements.

[1] Zero is a prime number.

[2] The ellipse with equation $\frac{(x+2)^2}{4}+\frac{(y+3)^2}{5}=1$ has area 20π .

[4] The distance between the three-dimensional points (5, 4, 2) and (3, 6, 1) is 3.

[8] If $f(x)=2\left(3x^{\frac{2}{3}}-5x\right)$, then $f(3)=-15+3\sqrt{3}$.

[16] The vector $<1,1>$ is a unit vector.

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Precalculus Team (no calculator)	**Question # 2**

A = the greatest solution to the equation $15-6x=x^2+24$

B = the radius of the circle with equation $x^2+62=87-y^2$

C = $\log_4 256$

D = $\sec^2\left(\frac{\pi}{4}\right)$

Find: $\left(\frac{\text{A}-\text{B}}{\text{C}+\text{D}}\right)^{-2}$

Middleton Invitational Precalculus Team (no calculator)	February 2006 Question # 3

Let $f(x) = \dfrac{x+3}{x-4}$.

A = $f\left(\dfrac{3}{2}\right)$ **B** = $\lim_{x \to -4}\left(f(x)\right)$ **C** = $f^{-1}(0)$

D = the *y*-intercept of the graph of $f(x)$

Find: $CD \div AB$.

Middleton Invitational Precalculus Team (no calculator)	February 2006 Question # 4

Let $f(x) = g(2x-1)$.

A = the period of *f*, when $g(x) = \tan x$

The equation ***x* = B** is the vertical asymptote of *f*, when $g(x) = \dfrac{1}{3x+4}$

The interval (**C,** ∞) is the domain of *f*, when $g(x) = \log_7 x$

Find: $\dfrac{120A}{\pi BC}$.

Middleton Invitational	February 2006
Precalculus Team (no calculator)	**Question # 5**

$f(x) = ax^5 + bx^4 - ax^3 - bx^2 + ax + b$, where *a* and *b* are positive integers.

Let **M** be the greatest real root of $f(x)$.

Let **N** be the leading coefficient of $f\left(-\frac{x}{a}\right)$.

Let **P** be the *y*-intercept of $\frac{f(x)+b}{a}$.

Let **Q** be $f(-1) + f(1)$.

Express the product **MNPQ** in terms of *a* and *b* without the use of negative exponents.

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Precalculus Team (no calculator)	**Question # 6**

Write the domain of each function given below in interval notation.

$f(x) = \sqrt{4-x}$ $\quad g(x) = \frac{4}{x}$ $\quad h(x) = \sqrt[3]{x^3 - 64}$ $\quad j(x) = \frac{1}{x^2 - 16}$

Let **A** be the number of times you wrote the union $(\cup)$ symbol.

Let **B** be the number of functions that include 4 in their domain.

Let **C** be the number of times you wrote $-\infty$.

Let **D** be the number of functions that include 0 in their domain.

Find **A + 10B + 100C + 0.1D** .

Middleton Invitational	February 2006
Precalculus Team (no calculator)	Question # 7

A = $\cos\theta$**,** if θ is the angle between the vectors <8,15> and $<-8,6>$

B = $\sin\theta$**,** if θ is the smallest angle in a right triangle with legs measuring 20 and 48.

C = $\cot\theta$**,** if $\sec\theta = \csc(2\theta)$ and $0 < \theta < \frac{\pi}{2}$.

D = $\tan\theta$**,** if θ is the acute angle between the *x*-axis and the line with equation $x - 3y = 4$.

Find the product **ABC²D.**

Middleton Invitational	February 2006
Precalculus Team (no calculator)	Question # 8

Each of the nine letters in the word **MIDDLETON** is written on a ball and placed in a bag.

W = the probability that two balls selected at random, without replacement, both have vowels on them.

X:Y = the odds that two balls selected at random, with replacement, have the same letter.

Z = The number of distinct linear arrangements of all the letters in **MIDDLETON**.

Find $\frac{WZ}{X^2Y}$.

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Precalculus Team (no calculator) | **Question # 9**

A = $\lim_{x\to 0}\frac{-\cos x+\tan x+1}{x}$

B = $\lim_{x\to 2}\frac{3x^3+x-6x^2-2}{4x^2-13x+10}$

C = $\lim_{x\to 3}(x)$

D = $\lim_{x\to\infty}\left(x^{-1}\right)$

Find: **4A + 3B + 2C + D**

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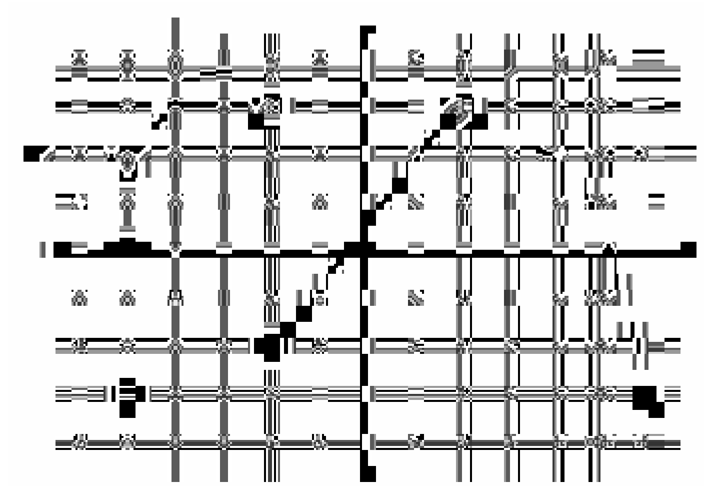
Precalculus Team (no calculator) | **Question # 10**

The function $f(x)$ is graphed to the right.
Marked points have integral coordinates.
Increments on both axes are 1.



A = The number of values of x at which $f(x)$ is discontinuous on the interval $(-\infty,0)$

B = $\lim_{x\to -5^+} f(x)$

C = $f(-2)$

D = The number of solutions to the equation $f^{-1}(x)=2$

E = The number of solutions to the equation $\frac{f(x)}{2}=1$

Find: **A + 3B + 5C + 7D + 11E**

Middleton Invitational	February 2006
Precalculus Team (no calculator)	Question # 11

Given that 3 is a root of the polynomial $b(x) = 10x^3 - 11x^2 - 72x + 45$:

Let **F** be the least root of *b*.

Let **G** be the greatest root of *b*.

Let **H** = $b(0.2)$.

Find: $H \cdot F^2 + 2G$.

Middleton Invitational	February 2006
Precalculus Team (no calculator)	Question # 12

The polar point $\left(-6, \frac{11\pi}{3}\right)$ has rectangular coordinates **(A, B)**.

The vertex of the parabola with polar equation $r = \frac{1}{3 - 3\cos\theta}$ has rectangular coordinates **(C, D)**.

The ellipse with equation $x^2 + 4y^2 - 4x + 8y = 0$ has eccentricity **E**.

Find: **2A + 4B + 6C + 8D + 10E .**

Middleton Invitational	February 2006
Precalculus Team (no calculator)	**Question # 13**

Find $\sin^2(0.5\theta)\cdot\cos(-2\theta)\cdot\csc^3\theta\cdot\cot(0.5\pi-\theta)$, when $\cos^2\theta = 0.36$ and θ is an angle in standard position with its terminal side in Quadrant I.

Middleton Invitational	February 2006
Precalculus Team (no calculator)	**Question # 14**

Let the vectors $A = <3, -1>$ and $B = <-3, 2>$.

$2A - 4B = <m, n>$

A is orthogonal to the vector $<5,\ p>$.

B is parallel to the vector $<5,\ t>$.

Find: $m + n - \dfrac{p}{t}$.

Find the number of distinct solutions to the equation $(\cos x)(\sin x) = 0$ on the interval $[0,10]$.

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Precalculus Team Solutions

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1. Zero is neither prime nor composite.
The ellipse in question has area $ab\pi = 2\pi\sqrt{5}$.
The distance between the points is indeed 3 ($\sqrt{2^2+2^2+1^2}$) $f(3) = 3(3)^{\frac{2}{3}} - 5(3) = -15 + 3\sqrt[3]{9}$.
Unit vectors have magnitude 1, and this vector has magnitude $\sqrt{1^2+1^2} = \sqrt{2}$.

$\frac{9}{16}$

2. **A = -3.** $x^2+6x+9=0 \rightarrow (x+3)^2=0 \rightarrow x=-3$
B = 5. $x^2+y^2=25 \rightarrow r=5$ **C = 4.** **D =** $\dfrac{1}{\cos^2\frac{\pi}{4}} = \dfrac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = 2$**.**

-10

3. $A = \dfrac{\frac{3}{2}+3}{\frac{3}{2}-4} = \dfrac{\frac{9}{2}}{-\frac{5}{2}} = -\dfrac{9}{5}$ $B = \dfrac{-4+3}{-4-4} = \dfrac{1}{8}$

$f^{-1}(0)$ is equal to the solution to $f(x)=0$, so $C=-3$. $D = f(0) = -\dfrac{3}{4}$.

-720

4. The period of $f(x)=\tan(2x-1)$ is $\dfrac{\pi}{\omega} = \dfrac{\pi}{2}$.
The asymptote of $f(x) = \dfrac{1}{3(2x-1)+4} = \dfrac{1}{6x+1}$ is $x = -\dfrac{1}{6}$.
The argument of a logarithm must be positive,
so the domain of $f(x)=\log_7(2x-1)$ is $\left(\dfrac{1}{2},\infty\right)$.

$\frac{4b^3}{a^6}$

5. Factor, and $f(x) = x^4(ax+b) - x^2(ax+b) + 1(ax+b) = (x^4-x^2+1)(ax+b)$. The first factor yields nonreal roots, so the only real root is $-\dfrac{b}{a}$.

The leading coefficient of $f\left(-\dfrac{x}{a}\right) = a\left(-\dfrac{x}{a}\right)^5 + \ldots = \dfrac{-1}{a^4}x^5 + \ldots$ is $-a^{-4}$.

The y-intercept of $\dfrac{f(x)+b}{a} = \ldots + \dfrac{b+b}{a} = \ldots + \dfrac{2b}{a}$ is $\dfrac{2b}{a}$.

$f(-1) = -a+b+a-b-a+b = -a+b$. $f(1) = a+b-a-b+a+b = a+b$.
$f(1)+f(-1) = -a+b+a+b = 2b$.

The product is $\left(-\dfrac{b}{a}\right)\left(-\dfrac{1}{a^4}\right)\left(\dfrac{2b}{a}\right)\left(\dfrac{2b}{1}\right) = \dfrac{4b^3}{a^6}$

433.3

6. $f(x):\ (-\infty,4]$ $\qquad g(x):\ (-\infty,0)\cup(0,\infty)$ $\qquad h(x):\ (-\infty,\infty)$
$j(x):(-\infty,-4)\cup(-4,4)\cup(4,\infty)$

A = 3, B = 3, C = 4, D = 3

$\frac{1}{17}$

7. $A=\frac{\vec{u}\bullet\vec{v}}{\|u\|\|v\|}=\frac{-64+90}{(17)(10)}=\frac{13}{85}$ $\qquad B=\frac{20}{52}=\frac{5}{13}$

$\sec\theta=\csc(2\theta)\ \rightarrow\ \frac{1}{\cos\theta}=\frac{1}{\sin(2\theta)}\ \rightarrow\ 2\sin\theta\cos\theta=\cos\theta\ \rightarrow\ \cos\theta(2\sin\theta-1)=0\ \rightarrow$

$\cos\theta=0$ *or* $\sin\theta=\frac{1}{2}$. Only $\frac{\pi}{6}$ works, so $C=\cot\theta=\sqrt{3}$.

$D=\tan\theta=m=\frac{1}{3}$.

$ABC^2D=\left(\frac{13}{85}\right)\left(\frac{5}{13}\right)\left(\frac{3}{1}\right)\left(\frac{1}{3}\right)=\frac{1}{17}$.

$\frac{135}{11}$

8. $W=\left(\frac{3}{9}\right)\left(\frac{2}{8}\right)=\frac{1}{12}$ $\qquad \left(\frac{2}{9}\right)\left(\frac{2}{9}\right)=\frac{4}{81}\ \rightarrow\ X:Y=4:77$

$Z=\frac{9!}{2!}=\frac{9!}{2}$

$$\frac{WZ}{X^2Y}=\left(\frac{1}{\cancel{12}}\right)\left(\frac{9\cdot\cancel{8}\cdot\cancel{7}\cdot\cancel{6}\cdot5\cdot\cancel{4}\cdot3\cdot\cancel{2}}{\cancel{2}}\right)\left(\frac{1}{\cancel{16}}\right)\left(\frac{1}{\underset{11}{\cancel{77}}}\right)=\frac{135}{11}$$

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9. $A=\lim\limits_{x\to0}\frac{-\cos x+\tan x+1}{x}=\lim\limits_{x\to0}\left(\frac{-\cos x+1}{x}\right)+\lim\limits_{x\to0}\left(\frac{\tan x}{x}\right)=0+1=1$

Let $f(x)=\frac{3x^3+x-6x^2-2}{4x^2-13x+10}=\frac{(3x^2+1)(x-2)}{(x-2)(4x-5)}=\frac{3x^2+1}{4x-5},\ x\neq2$.

Then $B=\lim\limits_{x\to2}f(x)=\frac{3(2)^2+1}{4(2)-5}=\frac{13}{3}$.

$C=3$

$D=\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)=0$.

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10. A = 2, B = 2, C = -2, D = 0, E = 3.

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11. $b(x)=(2x+5)(5x-3)(x-3)$. $F=-\frac{5}{2}$, $G=3$, $H=\frac{756}{25}$.

$-7+17\sqrt{3}$

12. $A=-6\cos\frac{11\pi}{3}=-3$; $B=-6\sin\frac{11\pi}{3}=3\sqrt{3}$

$$r=\frac{1}{3-3\cos\theta} \rightarrow 3r=1+3r\cos\theta \rightarrow 3\sqrt{x^2+y^2}=1+3x \rightarrow 9x^2+9y^2=9x^2+6x+1 \rightarrow$$

$$9y^2=6x+1 \rightarrow 9(y+0)^2=6\left(x+\frac{1}{6}\right) \rightarrow C=-\frac{1}{6};\ D=0$$

$$x^2-4x+4+\ 4(y^2+2y+1)\ =\ 8 \rightarrow \frac{(x-2)^2}{8}+\frac{(y+1)^2}{2}=1.\quad E=\frac{c}{a}=\frac{\sqrt{6}}{\sqrt{8}}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}.$$

$-\frac{7}{48}$

13. $\cos^2\theta=\frac{36}{100} \rightarrow \cos\theta=\frac{6}{10}=\frac{3}{5};\ \sin\theta=\frac{4}{5}\ ;\ \csc\theta=\frac{5}{4};\ \tan\theta=\frac{4}{3}$

$$\sin^2(0.5\theta)\cdot\cos(-2\theta)\cdot\csc^3\theta\cdot\cot(0.5\pi-\theta)=\left(\sqrt{\frac{1-\cos\theta}{2}}\right)^2\left(-\sin^2\theta+\cos^2\theta\right)\left(\csc^3\theta\right)\left(\tan\theta\right)=$$

$$-\left(\sqrt{\frac{\frac{2}{5}}{\frac{2}{1}}}\right)^2\left(\frac{16}{25}-\frac{9}{25}\right)\left(\frac{5}{4}\right)^3\left(\frac{4}{3}\right)=-\left(\frac{1}{\cancel{5}}\right)\left(\frac{7}{\cancel{25}}\right)\left(\frac{\cancel{125}}{64}\right)\left(\frac{4}{3}\right)=-\frac{7}{48}$$

14. $2A-4B\ =\ <6,-2>+<12,-8>\ =\ <18,-10> \rightarrow m=18,\ n=-10$

$<3,-1>\bullet<5,p>=0 \rightarrow 15-p=0 \rightarrow p=15$

$\frac{-3}{2}=\frac{5}{t} \rightarrow t=-\frac{10}{3}$.

$\frac{25}{2}$ (*or* 12.5)

$$m+n-\frac{p}{t}=18-10+\frac{\frac{15}{1}}{\frac{10}{3}}=\frac{25}{2}\ (or\ 12.5)$$

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15. $x=\frac{k\pi}{2}$ *on* $[0,10]\rightarrow x=\ 0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi,\frac{5\pi}{2},3\pi$