Algebra II

FELIX VARELA HIGH SCHOOL

FAMAT REGIONAL COMPETITION

FEBRUARY 4, 2006

Note: In all questions, NOTA stands for "None of the Above."

1. What is the equation of the directrix of the parabola $y^2+4y+8x+28=0$?

A. $x=-1$ B. $y=0$ C. $x=1$
D. $x=3$ E. NOTA

2. The geometric mean of two positive integers is 10. Which of the following could not be their sum?

A. 25 B. 29 C. 50
D. 101 E. NOTA

3. Let $f(x)=x^2+3$, $g(x)=4-x^3$, $h(x)=2x$ Evaluate $h(f(g(0)))$.

A. 0 B. 38 C. 39 D. 40 E. NOTA

4. Let $f(x)=x^2-12x+7$. What is the slope of the line containing the points $(-3,f(-3))$ and $(4,f(4))$?

A. $-5/7$ B. $-7/5$ C. 5
D. 11 E. NOTA

5. Let Ax+By=C and Dx+Ey=F be the equations of 2 distinct lines. If the 2 lines are parallel, which of the following must be true?

I. A=D II. B=E III. $C \neq F$

A. I only B. I, II only C. III only
D. I, II, III E. NOTA

6. The graphs of $f(x)=\log_3(x)$ and $g(x)=\log_7(x)$ intersect at a single point, (a,b). What is b/a?

A. 0 B. 1 C. 3
D. 7 E. NOTA

7. Let *r* be the line which passes through the points (-2,-4) and (4,14). Let *s* be the line which passes through the point (1,-1) and has slope m= -5. The lines *r* and *s* intersect at the point

A. (5/16,1/16) B. (35/16, -1/16)
C. (1/16, 35/16) D. (-1/16, 35/16)
E. NOTA

8. What is the area bounded by the graphs of $y=|x|$ and $y=-|3x|+8$?

A. 4 B. 8 C. 14 D.16 E. NOTA

9. Evaluate: $\begin{vmatrix} 3 & 4 & 0 \\ 2 & 8 & -5 \\ 1 & 0 & 9 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}$.

A. 104 B.124 C. 144
D. 164 E. NOTA

10. Let $x=4+\sqrt{3}$ be a zero of the polynomial function
$f(x)= x^2+ax+b$.
Then a+b =
A. 2 B. 3 C. 4 D. 5 E. NOTA

11. Solve for x to the nearest thousandth: $5^x=13$
A. 0.627 B. 0.628 C. 1.593
D. 1.594 E. NOTA

12. Let $f(x)=\sqrt{x^2+1}$. The domain of f(x) is the set of all real numbers, what is its range?
A. $(-\infty,\infty)$ B. $(-\infty,0]$
C. $[0, \infty)$ D. $[1, \infty)$
E. NOTA

13. Let z be a complex number such that z is a solution of the equation $x^3 -1 = 0$. Assume that $z \neq 1$. Then which of the following must be true?

I. z is a solution of the equation $x^2+x+1=0$
II. $\overline{z}$, the complex conjugate of z, is a solution to the equation $x^2+x+1=0$
III. z^2 is a real number

A. I only B. I,II only
C. I, III only D. I,I,III
E. NOTA

14. The two roots of
$f(x) = 3x^2 -2x +1$
are $a+bi$ and $c+di$. What is the distance between the points (a,b) and (c,d)?
A. $\frac{\sqrt{2}}{3}$ B. $\frac{2\sqrt{2}}{3}$ C. $\frac{2}{3}$
D. $\sqrt{2}$ E. NOTA

15. The function $g(x)=3x^2 + x^3 - x^7$ has
no negative real roots. How many non-real roots does it have?
A. 3 B. 4 C. 5 D. 6 E. NOTA

16. Which of the following describes the 4 roots of the function $f(x)=x^4-2x^3+7x-4$
A. 4 irrational
B. 2 irrational and 2 rational
C. 2 irrational real roots and 2 non-real
D. 2 rational and 2 non-real
E. NOTA

17. Which of the following functions passes through the origin?
A. $y=2e^{-x}+1$ B. $y=2e^{-2x}-1$
C. $y=e^{2x}-2$ D. $y=e^{2x}-1$
E. NOTA

18. The two circles
$(x+5)^2 + y^2 = 36$
and $(x+2)^2+y^2=9$
A. do not intersect.
B. intersect at exactly 1 point.
C. intersect at exactly 2 points.
D. Not enough information is given.
E. NOTA

19. What is the sum of the 2 real solutions of the equation
$e^{x+4}+e^0-e^{3x^2} = 1$?
A. –1/3 B. –1/4 C. 0
D. 1/4 E. NOTA

20. If you roll 3 dice, what is the probability that the total will be 5? Assume that you are using standard, 6-sided dice, and that all die are fair (that is, each face has an equal probability of coming up).
A. 1/108 B. 1/36 C. 1/52
D. 5/32 E. NOTA

21. Let a>1 be a fixed, positive real number. Then what is the sum of
$\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^4}+\frac{1}{a^6}+\ldots$
A. $\frac{a}{a^2-1}$ B. $\frac{1}{a-1}$
C. $\frac{a^2+a-1}{a^3-a}$ D. $\frac{1}{a^2-1}$ E. NOTA

22. Simplify: $\frac{4+i}{-6+4i}$
A. $\frac{-10-11i}{26}$ B. $\frac{-10+11i}{26}$
C. $\frac{-14-11i}{26}$ D. $\frac{-14+11i}{26}$

E. NOTA

23. A given nondegenerate conic has eccentricity e=1.38. Which of the following must be true?
A. The conic is an ellipse
B. The conic is a hyperbola
C. Eccentricity cannot be greater than 1
D. The type of conic cannot be determined solely by the eccentricity
E. NOTA

24. Let $f(x)=9x^7+3x^5-6$. Which of the following is not a possible root of f(x)?
A. –1 B. 2/3 C. 1/3
D. 9 E. NOTA

25. Consider the following system of equations: ax+by+cz=d, ex+fy+gz=h, and ix+jy+kz=m, where a,b,c,d,e,f,g,h,i,j,k and m are all constants. Let $Q=\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix} \neq 0$.
Then y =

A. $Q\begin{vmatrix} a & b & d \\ e & f & h \\ i & j & m \end{vmatrix}$ B. $Q\begin{vmatrix} a & d & c \\ e & h & g \\ i & m & k \end{vmatrix}$

C. $\dfrac{\begin{vmatrix} a & b & d \\ e & f & h \\ i & j & m \end{vmatrix}}{Q}$ D. $\dfrac{\begin{vmatrix} a & d & c \\ e & h & g \\ i & m & k \end{vmatrix}}{Q}$

E. NOTA

26. Thirty people walk by a donation box. Each person drops money in the box, and each person gives \$12 more than the previous person. The last person gives a \$400 donation. How much money is in the box after all 30 people have donated (assume the box was initially empty)?
A. \$6770 B. \$6780 C. \$6790
D. \$6800 E. NOTA

27. Which of the following is a solution of the equation $\begin{vmatrix} 6-x & -3 \\ 1 & 2-x \end{vmatrix}=0$?
A. 0 B. 1 C. 2 D. 3 E. NOTA

28. What is the radius of the circle $x^2+y^2+6x-4y=563$ (to the nearest tenth)?
A. 23.7 B. 24.0
C. 24.1 D. 25.2 E.NOTA

29. Let $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then
$A^{58}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
for some real integers a,b,c and d. What is a+b+c+d?
A. 58 B. 59 C. 6084
D. 732489 E. NOTA

30. If A,B,C are all non-zero, n by n matices, which of the following are true?
I. (AB)C=A(BC)
II. A(B+C)=AB+AC
III. AB=BA
A. I, II only B. II,III only C. I only
D. I,II,III E. NOTA

March Algebra 2 Individual Solutions

1. $y^2+4y+8x+28=0$ means $(y+2)^2= -8(x+3)$. The vertex is (-3, -2), a=2. x = -1 A
2. $\sqrt{xy} = 10$ so xy=100. The possibilities are (1,100), (2,50), (4,25), (5,20), (10,10) C
3. h(f(g(0)))=h(f(4))=h(16+3)=h(19)=38. B
4. f(-3)=52, f(4)=-25, m=(52+25)/(-3-4)=77/-7=-11 E
5. We can rewrite the equations as: y=C/B – A/Bx and y=F/E – D/Ex. We must have D/E=A/B, but not necessarily I or II. To be distinct, we need C/B≠F/E, not III. E
6. All logarithmic graphs pass through the point (1,0). 0/1=0. A
7. *r* has equation y=3x+2, and *s* has equation y= -5x+4. When 3x+2=-5x+4, x=1/4. E
8. They intersect at (2,2) and (-2,2). Considered as 2 triangles, A=2(1/2)(8)(2)=16 D
9. The 1st determinant,(3)(8)(9)+(4)(-5)(1)+(0)(2)(0)-(1)(8)(0)-(0)(-5)(3)-(9)(2)(4)=124 B
10. $f(x=4+\sqrt{3})=0=(4+\sqrt{3})^2+a(4+\sqrt{3})+b=(19+4a+b)+(a+8)\sqrt{3}$. So a=-8 and b=13. D
11. $5^x=13$ so $x = \log_5(13) = \log(13)/\log(5) = 1.594$ D
12. The range of $g(x)=x^2+1$ is $[1, \infty)$. The range of $f(x)=\sqrt{x^2+1}$ is also $[1, \infty)$ D
13. The equation $x^3 -1 = (x-1)(x^2+x+1)=0$ has 3 complex roots. We know that $z\neq1$, so z must be a root of x^2+x+1 . I is true. II is true because a number's complex conjugate is always another root of a polynomial with real coefficients. III is false because z is non-real, so if z^2 were real then $zz^2=z^3=1$ would be non-real. B
14. The roots of f(x) are $(1+i\sqrt{2})/3$ and $(1 - i\sqrt{2})/3$. $d=2\sqrt{2}/3$ B
15. g(x) has a double root at 0. $h(x)=3+x+x^5$ crosses the x-axis somewhere, but has no negative real roots and by Descartes' Sign Rule, it has at most 1 positive real root. B
16. $x^4-2x^3+7x-4 = (x^2-3x+4)(x^2+x-1)$ The discriminants of these are –7 and 5. C
17. $y=e^{2(0)}-1=1-1=0$ D
18. The two circles described are internally tangent, and intersect at the point (1,0). B
19. $e^{x+4}+1-e^{3x^2} = 1$ so $e^{x+4} = e^{3x^2}$, and $3x^2=x+4$ The solutions are x=4/3 and x=-1.E
20. To get a 5, you need to roll a 1,1,3 or a 1,2,2. The probability of rolling either one is $(1/6)^3(3)=1/72$, and 2(1/72)=1/36. B
21. $\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^4}+\frac{1}{a^6}+\ldots=\frac{1}{a}+\sum_{i=1}^{\infty}\frac{1}{a^{2i}}=(\frac{1}{a^2})/(1-\frac{1}{a^2}) = \frac{a^2+a-1}{a^3-a}$. C
22. $\frac{4+i}{-6+4i}\cdot\frac{-6-4i}{-6-4i}=\frac{-10-11i}{26}$ A
23. If the eccentricity is greater than 1, then the conic is a hyperbola. B
24. $9x^7+3x^5-6=3(3x^7+x^5-2)$, so the possible roots are +/-1, +/-2, +/-1/3, +/-2/3. D
25. This is a direct application of Cramer's Rule. D
26. The first person gave 400+(-12)(29)=52 ,Total=(1/2)30(400+52)=6780 B
27. $\begin{vmatrix} 6-x & -3 \\ 1 & 2-x \end{vmatrix}=(6-x)(2-x)-(-3)=12-8x+x^2+3=x^2-8x+15=(x-3)(x-5)=0$ D
28. $x^2 + y^2+6x-4y-563=x^2+6x+9+y^2-4y+4-576=(x+3)^2+(y-2)^2-576=0$ $\sqrt{576}=24$ B
29. Note that $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and in general $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. a+b+c+d=1+58+0+1=60 E
30. Matrix multiplication is associative and distributive, but not commutative. A

March Algebra II Team Questions

Question 1

Define an operation # as follows: a # b = ab + a + b.

Find (2 # 1) # (1 # 0) + (1 # 3)(2 # 3)(3 # 3) + (4 # –1)(3 # 0)(4 # 0)(5 # –5)

Question 2

The lines $y = -2$, $y = x-1$, $y = -2x+20$ and $y = -\frac{1}{3}x+18$ intersect each other at the points R,S,T,P,Q, and V. Find the sum of the x-coordinates of these points.

Question 3

Let A be the number of non-real roots of the function $f(x)=x^2+3x-2$.

Let B be the number of real roots of the function $g(x)=x^3-3x^2+5$.

Let C be the number of real roots of the function $h(x)=x^4+13x^2-21x+9$.

Let D be the number of non-real roots of the function $j(x)=x^7+x^6+x^5+x^4+x^3+x^2+x+1$.

Find A+B+C+D

Question 4

List the letters of all true statements.

A. If f(x) is a polynomial function with real coefficients and *a* is a root of f(x), then $\overline{a}$, the complex conjugate of *a*, is a root of f.

B. If f(x) is any polynomial function of degree greater than zero, then there exists a complex number *z*, such that f(*z*)=0.

C. If f(x) is a polynomial function of degree greater than or equal to 2, and *a* and *b* are real, rational numbers such that f(*a*) = f(*b*) = 0, then $a \neq b$.

D. If g(x) is a polynomial function with at most one real root, then there exists a polynomial function f(x) with real coefficients such that the domain of the function $h(x)=\frac{f(x)}{g(x)}$ is $(-\infty,\infty)$.

Question 5

A bag contains 52 marbles of various colors: 13 red, 13 blue, 13 green and 13 orange. You reach into the bag and randomly choose 5 marbles. What is the probability that all of the marbles you have chosen are of the same color? (round your answer to the nearest thousandth)

Question 6

When working with complex numbers, it is sometimes possible to "factor" prime integers. For example, $5=(2+i)(2-i)$. The numbers 13, 17, 29 and 37 can also be "factored" in this way. Let $13=(a+bi)(a-bi)$, $17=(c+di)(c-di)$, $29=(e+fi)(e-fi)$ and $37=(g+hi)(g-hi)$, where $a < b$, $c < d$, $e < f$ and $g < h$ and where a,b,c,d,e,f,g and h are all positive integers. Find a+b+c+d+e+f+g+h.

Question 7

Let $f(x)=|x-3x^2|$, $g(x)=\frac{x}{10x+3}$, $h(x)=x+9$. Let A=f(h(f(0))), B=h(f(h(-1))) and C=g(h(f(2))). Find A+BC.

Question 8

Let $A = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$ Let $B = \frac{1}{9}+\frac{1}{81}+\frac{1}{729}+\ldots$ Let $C = \frac{1}{27}+\frac{1}{243}+\frac{1}{2187}+\ldots$

Find A+B+C

Question 9

Let A(-2,$2\sqrt{3}$), B(1,$-\sqrt{15}$) and C(0,4) be 3 points in the same plane. Find the distance between the midpoints of $\overline{AB}$ and $\overline{BC}$ to the nearest hundredth.

Question 10

Ana is playing a game with Bob. Bob deals 3 cards, facedown, of which 2 are black and 1 is red. Ana will win the game if she correctly chooses the red card. When Bob asks her which card she wants to choose, she selects the first card. Without revealing whether or not she is correct, Bob turns over the second card and shows that it is black. He offers Ana the choice to change her original guess (and choose the third card) or stick with her original choice. Ana decides to switch. What is the probability that she will win the game?

Question 11

Let $A=13^{\log_{13}(e^4)}$, $B=\ln(e^{\log_{13}(4)})$, $C=\log_{10}(e^{\ln(100)})$ and $D=e^{3\ln(4)}$. Find ln(AB)+ln(CD) to the nearest thousandth.

Question 12

What is the 3rd number on the 128th line of Pascal's Triangle? The first line of Pascal's Triangle is the single digit 1.

Question 13

What is the area of the region bounded by the x-axis, the y-axis, and the lines y = -3x+6 and x=7-y?

Question 14

The characteristic polynomial of a matrix A is given by the function $f_A(x)=\det(A-Ix)$, where det(X) is the determinant of the matrix X, and I is the identity matrix. Find the coefficient of the linear term of the function $f_A(x)$ where $A=\begin{vmatrix} 3 & 4 & -1 \\ 0 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix}$ and

$I=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Question 15

Write the letters of all true statements in alphabetical order. In all of the following, let R, S and T be three points that do not lie on the same line. Also, all figures should be assumed to lie in the same plane.

A. There is exactly one circle that contains the points R,S and T.

B. There is exactly one parabola that contains the points R,S and T.

C. If R lies on a parabola, then the distance from R to the focus is the same as the distance from R to the directrix.

D. If R and S lie on an ellipse, then the distance from R to the nearest focus point is less than the distance from S to either focus point.

E. The circumference of the circle $x^2+y^2=9$ is 9π

F. If the length of the semimajor axis of an ellipse is a, and the length of the semiminor axis is b, then the eccentricity of the ellipse is $\sqrt{1-\frac{b^2}{a^2}}$.

March Algebra 2 Team Solutions

1. (2 # 1) # (1 # 0) + (1 # 3)(2 # 3)(3 # 3) + (4 # –1)(3 # 0)(4 # 0)(5 # –5) = 5 # 1 +(7)(11)(15) + (-1)(3)(4)(-25)=11+1155+300 = **1466.**

2. The 4 lines intersect each other at 6 different points: (6/5,88/5), (57/4,53/4), (7,6), (-1,-2), (11,-2) and (60,-2). The sum of the x coordinates is **92.45 or 1849/20.**

3. The discriminant of f(x) is 17, so it has no non-real roots. g(x) can be easily graphed, and has 1 real root. h(x) doesn't cross the x-axis, so it has no real roots. j(x) has 6 non-real roots. 0+1+0+6=**7**.

4. A is true, B is a statement of the Fundamental Theorem of Algebra, also true. C is false, f(x) might have a double root. D is also false, if g(x) has a root, h(x) will have (at best) a point-discontinuity. **A,B.**

5. The number of ways to have 5 marbles of the same color is 13 choose 5, times 4 (because there are 4 different colors) = 5148. The total number of ways to choose the 5 marbles is 52 choose 5, or 2568960. Thus, the probability is **0.002.**

6. 13=(2+3i)(2-3i), 17==(4+i)(4-i), 29==(5+2i)(5-2i) and 37=(6+i)(6-i). 5+5+7+7=**24.**

7. A=f(h(0))=f(9)=234. B=h(f(8))=h(184)=193. C=g(h(10))=g(19)=19/193. 234+19=**253**.

8. A=(1/3)/(1-1/3)=1/2. B+C=A-1/3, so A+B+C=2A-1/3=2(1/2)-1/3=**2/3.**

9. The midpoint of $\overline{AB}$ is $(\frac{-1}{2}, \frac{(2\sqrt{3}-\sqrt{15})}{2})$ and the midpoint of $\overline{BC}$ is $(\frac{1}{2}, \frac{4-\sqrt{15}}{2})$. So the distance between them is $\sqrt{1^2+(\frac{4-2\sqrt{3}}{2})^2}$= **1.04**

10. Ana has a 1/3 chance of choosing correctly on her first guess. So the probability that the red card is among the 2 she didn't choose is 2/3. If the card is among those two, Bob has no choice in which card he reveals (that is, there is a 2/3 chance that the red card is card 3). So the probability that she will win the game after switching is **2/3**.

11. A=e^4, B= $\log_{13}(4)$, C=2, D=$e^{3\ln(4)}$. So ln(AB)+ln(CD)=ln(A)+ln(B)+ln(C)+ln(D)= 4 + -0.615 + 0.693 + 4.159 = **8.237**

12. Looking at the first few lines of the triangle, it is clear that the 3rd number on the n-th row is the sum of the first n-2 integers. 1+2+...+126=1/2(126)(127)=8001. Alternately, this can be calculated by binomial expansion – the 3rd number in the 128th line would be 127 nCr 2 = **8001.**

13. The area can be determined as the difference of area of two triangles: 1/2(7)(7)-1/2(2)(6)=24.5-6=**18.5 or 37/2.**

14. $\begin{vmatrix} 3-x & 4 & -1 \\ 0 & 1-x & 2 \\ 2 & 1 & 2-x \end{vmatrix} = -x^3+6x^2-11x+18$. **-11.**

15. A is true, as was seen in problem 9. B is false, 3 points uniquely define TWO parabolas. C is true, this is the definition of a parabola. D is false, there is no reason it would be true. E is false, the circumference is 6π. F is true, it is the definition of eccentricity for an ellipse. **A,C,F.**

March Algebra 2 Individual Answers
1. A
2. C
3. B
4. E
5. E
6. A
7. E
8. D
9. B
10. D
11. D
12. D
13. B
14. B
15. B
16. C
17. D
18. B
19. E
20. B
21. C
22. A
23. B
24. D
25. D
26. B
27. D
28. B
29. E
30. A

March Algebra 2 Team Answers
1. 1466
2. 92.45 or 1849/20
3. 7
4. A,B
5. 0.002
6. 24
7. 253
8. 2/3
9. 1.04
10. 2/3
11. 8.237
12. 8001
13. 18.5 or 37/2
14. -11
15. A,C,F