Ideal and Irrational Numbers 12

The Pythagoreans were followers of **Pythagoras** (572–497 B.C.), the philosopher and mathematician best known for the famous right triangle theorem named after him ($a^2 + b^2 = c^2$). The Pythagoreans believed that all of nature could be explained by numbers. In fact, their motto stated: "All things are numbers." They especially valued the numbers 1, 2, 3, and 4, which they called the *tetractys*. The Pythagorean oath was: "I swear in the name of the *Tetractys*, which has been bestowed on our soul." They saw fourness in many things, including the four geometric elements (point, line, surface, and solid) and the four material elements (earth, air, fire, and water).

Because the sum of the four numbers of the *tetractys* is 10, the Pythagoreans considered 10 to be the ideal number. They went to great lengths to build an astronomical theory based on number relationships. For example, they maintained that 10 represented the universe, so there must be ten bodies in the heavens. The earth, the sun, the moon, and the five then-known planets made up eight of these bodies. To complete their universe, they introduced a "central fire" and a "counter-earth." Humans couldn't see these two bodies because the portion of the earth where they lived faced away from the bodies.

Pythagoras stressed the importance of both whole numbers and the ratio of whole numbers in the study of nature. After his death, however, his followers used the method of indirect proof to establish that $\sqrt{2}$ cannot be written as the ratio of two whole numbers. In doing so, they discovered irrational numbers—numbers that can't be written as the ratio of two integers.

This discovery caused considerable consternation among the Pythagoreans, because many of Pythagoras' widely accepted conclusions were based on the implicit assumption that all numbers could be written as the ratio of two whole numbers. So as not to discredit him, the Pythagoreans took pains to keep their discovery a secret. Legend has it that a Pythagorean named Hippasus revealed the secret to outsiders. As a result he was tossed overboard by his fellow Pythagoreans while at sea.

For more on irrational numbers, see vignettes 19, 33, and 52. ★

The Women Pythagoreans

During a time when women were considered intellectually inferior to men, the Pythagorean community welcomed women as equals, providing them with opportunities to participate in the fields of science and mathematics. Pythagoras, known as the Feminist Philosopher, encouraged women as students and as teachers. Toward the end of his life, he married one of his more successful students, Theano. An accomplished cosmologist and healer, Theano headed the community after his death, and although she and her daughters faced political persecution, they continued to spread Pythagorean philosophies throughout Greece and Egypt.

Detail from Gregor Reisch's* Margarita philosophica *(1503), showing Pythagoras working with arithmetic.

Activities

1. Two of the most interesting numbers in the real number system are π and e. What were the processes used to establish that both of these numbers are irrational?

2. Provide a counterexample to prove that each statement is false.
   a. The sum of two irrational numbers is irrational.
   b. The difference of two irrational numbers is irrational.
   c. The product of two irrational numbers is irrational.
   d. The quotient of two irrational numbers is irrational.

3. A Pythagorean triple is a set of integers a, b, and c that could be the sides of a right triangle. That is, $a^2 + b^2 = c^2$. Show that the following method always generates Pythagorean triples but does not generate *all* Pythagorean triples: Square an odd integer greater than 1. Find the two consecutive integers whose sum is equal to the square of the chosen number. The integer you squared and the consecutive integers you found form a Pythagorean triple.

4. What did the mathematicians of ancient Greece find interesting about the number 8?

5. Prove that if a, b, and c are numbers satisfying the Pythagorean relation $a^2 + b^2 = c^2$, then, for any constant k, the numbers ka, kb, and kc will also satisfy the relation.

Related Reading

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