The Paradoxes of Zeno

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The ancient Greeks were masters of geometry, but they shied away from the concept of the infinite. No one was more influential in promoting a suspicion of the infinite than **Zeno of Elea** (ca 450 B.C.). Little is known of Zeno's life other than that he loved controversy. This trait is evident in his mathematical paradoxes, which assert that motion is impossible and that certain frequently observed events can't happen.

The *Dichotomy*

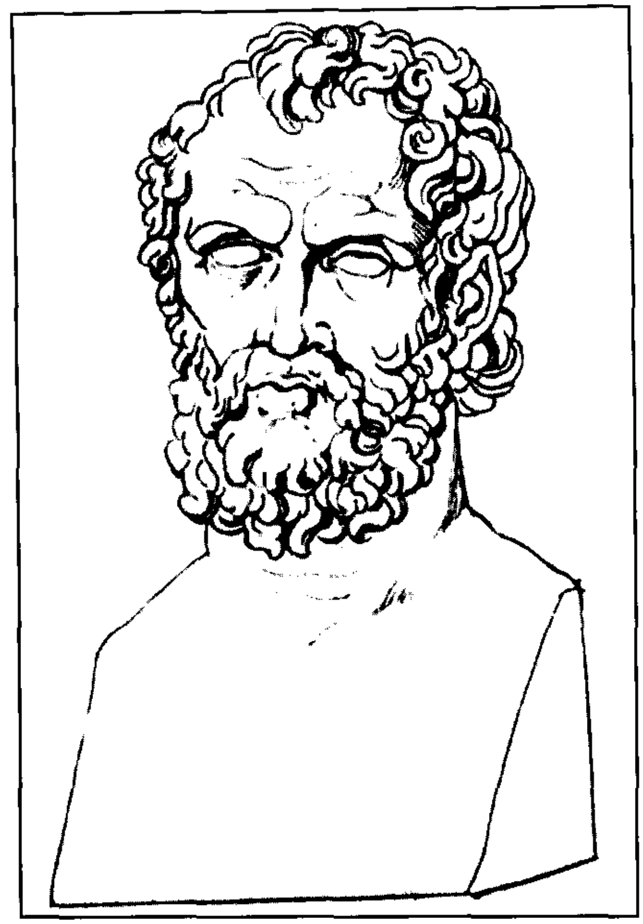
> *Before an object can travel a given distance, it first must travel half this distance; but before it can do this, it must travel the first quarter of the distance; and before this, the first eighth of the distance, and so on, ad infinitum. Since it is impossible to exhaust this infinite collection of events, motion cannot happen.*

If we'd never perceived an object in motion, this paradox might convince us that it's impossible to hit a target with a bow and arrow—the arrow wouldn't travel any distance.

The *Achilles*

> *Achilles is racing against a tortoise. Since the tortoise is slower, it is given a head start. Once the race starts, Achilles cannot catch the tortoise because when he reaches the initial position of the tortoise, it will have moved on. Continuing this line of reasoning, whenever Achilles gets to where the tortoise once was, the tortoise has advanced somewhat farther on. Hence, Achilles cannot overtake the tortoise.*

The Greeks knew that objects, such as arrows, can travel a certain distance. They knew that a fast runner can overtake a slower runner who may be temporarily ahead in a race. Try as they might, though, they couldn't resolve Zeno's paradoxes. As a result, they developed a deeply rooted suspicion of the infinite and the infinitesimal.



Engraved nineteenth-century illustration of Zeno, based on a bust found in a Roman villa. His deeply furrowed brow no doubt attempts to illustrate his love of paradox.

Zeno's arguments perplexed mathematicians for centuries. The confusion the paradoxes generated was compounded by the fact that infinity was often considered to have the property of a very large number. Swiss mathematician **Leonhard Euler** (1707–1783) did not hesitate to say that $1/0$ is infinite and that $2/0$ is twice as large as $1/0$. It was not until the nineteenth century, when **Georg Cantor** (1845–1918) developed his ideas about set theory and the theory of the infinite, that the mysteries of Zeno's paradoxes were solved.

For more on infinity, see vignettes 14, 51, 75, and 79. ★

Activities

1. Analyze this argument put forth by the Greek philosopher **Eubulides:** One grain of sand does not make a pile, and, if we add another grain of sand to the one, the two grains do not make a pile. When a grain of sand is added to a nonpile, this still doesn't make a pile. Hence, it is impossible to have a pile of sand.

2. Suppose Achilles gave the tortoise a head start of 1,000 meters and that he could run ten times faster than the tortoise. Let the length of the racetrack be x meters. What must be true about the value of x if the following is true?

   a. It is possible for Achilles to win the race.

   b. It is impossible for Achilles to win the race.

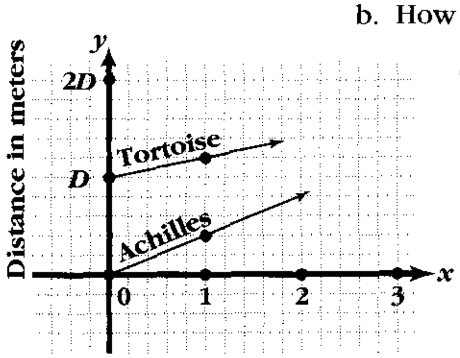
   c. The race ends in a tie.

3. In the figure, the tortoise starts D meters ahead of Achilles.

   a. How much faster is Achilles than the tortoise?

   b. How long will it take Achilles to catch the tortoise?

   c. In terms of D, how far will Achilles be from his starting point when he catches up to the tortoise?



Related Reading

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