INFINITUDE OF PRIMES

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> *This further is observable in number, that it is that which the mind makes use of in measuring all things that by us are measurable, which principally are expansion and duration; and our idea of infinity, even when applied to those, seems to be nothing but the infinity of number. For what else are our ideas of Eternity and Immensity, but the repeated additions of certain ideas of imagined parts of duration and expansion, with the infinity of number; in which we can come to no end of addition?*
>
> —John Locke, An Essay concerning Human Understanding

Prime numbers—positive integers greater than 1 that are evenly divisible only by themselves and by 1—have been a source of fascination for mathematicians through the ages because they are considered the building blocks from which all integers are made. We credit **Euclid** (ca 300 B.C.) with an elegant proof establishing that the number of prime numbers is infinite. Influenced by the Pythagoreans, he used the method of indirect proof.

Suppose that there are only a finite number of primes,

$p_1, p_2, p_3, \ldots, p_k,$

where k is a positive integer representing the number of primes. Construct a number P that is one greater than the product of the numbers in the set of primes. That is,

$P = p_1 \times p_2 \times p_3 \times \ldots \times p_k + 1.$

Since P is clearly greater than any of the primes in the list, it must be composite since the list contains all prime numbers. Since every composite number is a product of primes, there must be a prime number that divides P. But division by any of the primes $p_1, p_2, p_3, \ldots, p_k$ results in a remainder of 1, so none of these primes is a factor of P. Hence there must be a prime number that is not one of the finite collection $p_1, p_2, p_3, \ldots, p_k$. This contradicts the assumption that this set contains all of the primes. Hence, the assertion that there is only a finite number of primes is false. We conclude that the number of prime numbers is infinite.

Because Euclid proved that there is no such thing as the largest prime number, we don't bother to search for it. However, some mathematicians use modern computers to periodically find a number that temporarily holds the title of "the largest known prime number." Before the invention of computers, finding the largest known prime presented quite a challenge. One number that held this distinction for 75 years was discovered by French mathematician **Edouard Anatole Lucas** in 1876. The number is:

$2^{127} - 1 = 170,141,183,460,469,231,731,687,303,715,884,105,727.$

For more on infinity, see vignettes 13, 51, 75, and 79. ★

ACTIVITIES

1. **Christian Goldbach** (1690–1764) made a conjecture about prime numbers—it remains unproven to this day. What is Goldbach's conjecture?

2. What is Bertrand's conjecture (made in 1845) about prime numbers?

3. Pairs of primes that differ by two—such as {3, 5}, {5, 7}, and {11, 13}—are called twin primes. What is historically interesting about twin primes?

4. For over 2,000 years, mathematicians have sought to find a polynomial function $f(k)$ which, for values $k = 1, 2, 3, \ldots$, would yield only prime numbers. One function that starts off well is $f(k) = k^2 + k + 41$. Test this function by calculating $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$. Can you find a positive integer less than 50 for k such that $f(k)$ is not a prime? (If you approach this problem algebraically, there is one obvious value!)

5. If $f(k) = k^2 - 79k + 1601$, then $f(k)$ is a prime for $k = 1, 2, 3, \ldots, N$. Find the value for N such that $f(N)$ is a prime but $f(N + 1)$ is not.

6. Consider numbers formed by taking the product of consecutive primes and then adding 1. That is, consider the sequence $2 + 1, 2 \times 3 + 1, 2 \times 3 \times 5 + 1, 2 \times 3 \times 5 \times 7 + 1, 2 \times 3 \times 5 \times 7 \times 11 + 1, \ldots$ What is the first number in this sequence that is not prime?

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