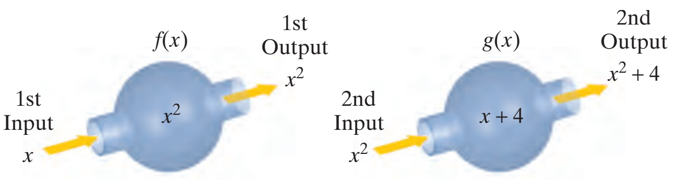
109. I have two functions. Function f models total world population x years after 2000 and function g models population of the world's more-developed regions x years after 2000. I can use $f - g$ to determine the population of the world's less-developed regions for the years in both function's domains.

110. I must have made a mistake in finding the composite functions $f \circ g$ and $g \circ f$, because I notice that $f \circ g$ is not the same function as $g \circ f$.

111. This diagram illustrates that $f(g(x)) = x^2 + 4$.



In Exercises 112–115, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

112. If $f(x) = x^2 - 4$ and $g(x) = \sqrt{x^2 - 4}$, then $(f \circ g)(x) = -x^2$ and $(f \circ g)(5) = -25$.

113. There can never be two functions f and g, where $f \neq g$, for which $(f \circ g)(x) = (g \circ f)(x)$.

114. If $f(7) = 5$ and $g(4) = 7$, then $(f \circ g)(4) = 35$.

115. If $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$, then $(f \circ g)(5) = g(2)$.

116. Prove that if f and g are even functions, then fg is also an even function.

117. Define two functions f and g so that $f \circ g = g \circ f$.

Preview Exercises

Exercises 118–120 will help you prepare for the material covered in the next section.

118. Consider the function defined by

$$\{(-2, 4), (-1, 1), (1, 1), (2, 4)\}.$$

Reverse the components of each ordered pair and write the resulting relation. Is this relation a function?

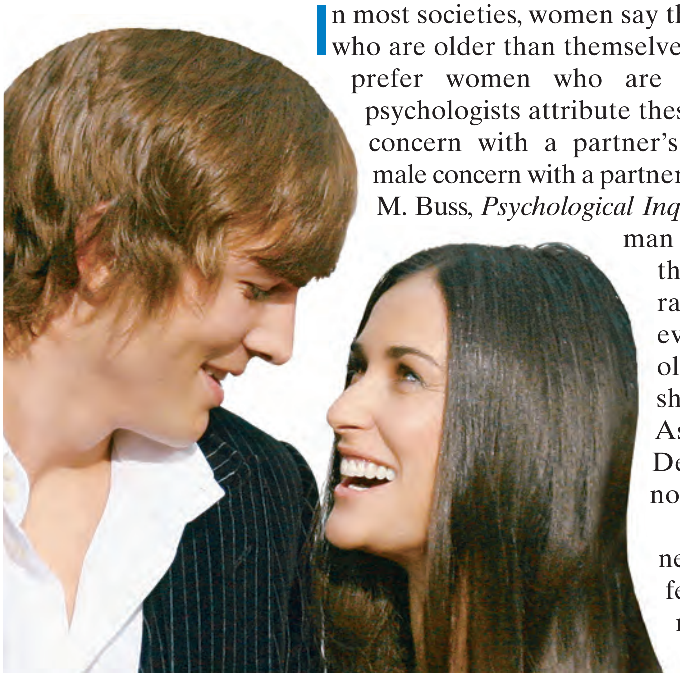
119. Solve for y: $x = \frac{5}{y} + 4$.

120. Solve for y: $x = y^2 - 1, y \geq 0$.

Section 1.8 Inverse Functions

Objectives

1. Verify inverse functions.
2. Find the inverse of a function.
3. Use the horizontal line test to determine if a function has an inverse function.
4. Use the graph of a one-to-one function to graph its inverse function.
5. Find the inverse of a function and graph both functions on the same axes.



In most societies, women say they prefer to marry men who are older than themselves, whereas men say they prefer women who are younger. Evolutionary psychologists attribute these preferences to female concern with a partner's material resources and male concern with a partner's fertility (*Source:* David M. Buss, *Psychological Inquiry*, 6, 1–30). When the man is considerably older than the woman, people rarely comment. However, when the woman is older, as in the relationship between actors Ashton Kutcher and Demi Moore, people take notice.

Figure 1.65 on the next page shows the preferred age difference in a mate in five selected countries.

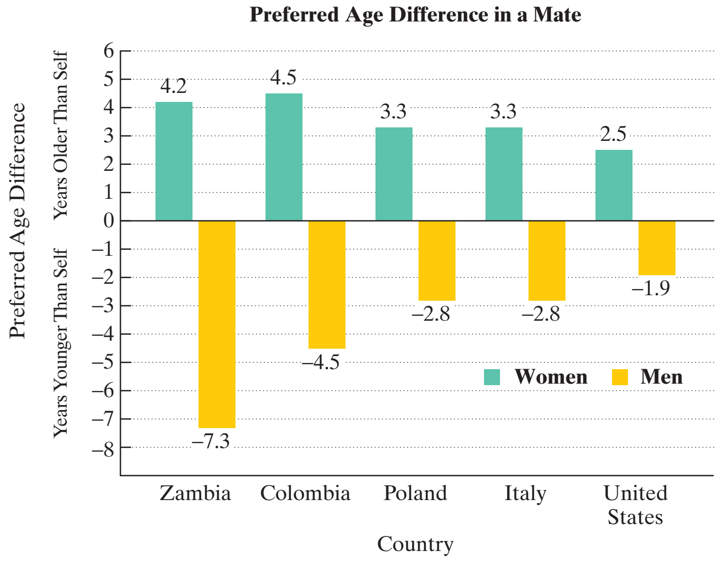


Figure 1.65
Source: Carole Wade and Carol Tavris, *Psychology*, 6th Edition, Prentice Hall, 2000

We can focus on the data for the women and define a function. Let the domain of the function be the set of the five countries shown in the graph. Let the range be the set of the average number of years women in each of the respective countries prefer men who are older than themselves. The function can be written as follows:

f: {(Zambia, 4.2), (Colombia, 4.5), (Poland, 3.3), (Italy, 3.3), (U.S., 2.5)}.

Now let's "undo" f by interchanging the first and second components in each of its ordered pairs. Switching the inputs and outputs of f, we obtain the following relation:

Same first component

Undoing f: {(4.2, Zambia), (4.5, Colombia), (3.3, Poland), (3.3, Italy), (2.5, U.S.)}.

Different second components

Can you see that this relation is not a function? Two of its ordered pairs have the same first component and different second components. This violates the definition of a function.

If a function f is a set of ordered pairs, (x, y), then the changes produced by f can be "undone" by reversing the components of all the ordered pairs. The resulting relation, (y, x), may or may not be a function. In this section, we will develop these ideas by studying functions whose compositions have a special "undoing" relationship.

Inverse Functions

Here are two functions that describe situations related to the price of a computer, x:

$$f(x) = x - 300 \qquad g(x) = x + 300.$$

Function f subtracts \$300 from the computer's price and function g adds \$300 to the computer's price. Let's see what $f(g(x))$ does. Put $g(x)$ into f:

$$f(x) = x - 300$$

This is the given equation for f.

Replace x with $g(x)$.

$$\begin{aligned} f(g(x)) &= g(x) - 300 \\ &= x + 300 - 300 \\ &= x. \end{aligned}$$

Because $g(x) = x + 300$, replace $g(x)$ with $x + 300$.

This is the computer's original price.

Using $f(x) = x - 300$ and $g(x) = x + 300$, we see that $f(g(x)) = x$. By putting $g(x)$ into f and finding $f(g(x))$, the computer's price, x, went through two changes: the first, an increase; the second, a decrease:

$$x + 300 - 300.$$

The final price of the computer, x, is identical to its starting price, x.

In general, if the changes made to x by a function g are undone by the changes made by a function f, then

$$f(g(x)) = x.$$

Assume, also, that this "undoing" takes place in the other direction:

$$g(f(x)) = x.$$

Under these conditions, we say that each function is the *inverse function* of the other. The fact that g is the inverse of f is expressed by renaming g as f^{-1}, read f-inverse. For example, the inverse functions

$$f(x) = x - 300 \qquad g(x) = x + 300$$

are usually named as follows:

$$f(x) = x - 300 \qquad f^{-1}(x) = x + 300.$$

We can use partial tables of coordinates for f and f^{-1} to gain numerical insight into the relationship between a function and its inverse function.

Computer's regular price

$300 reduction

x	$f(x) = x - 300$
1200	900
1300	1000
1400	1100

Ordered pairs for f: (1200, 900), (1300, 1000), (1400, 1100)

Price with $300 reduction

$300 price increase

x	$f^{-1}(x) = x + 300$
900	1200
1000	1300
1100	1400

Ordered pairs for f^{-1}: (900, 1200), (1000, 1300), (1100, 1400)

The tables illustrate that if a function f is the set of ordered pairs (x, y), then its inverse, f^{-1}, is the set of ordered pairs (y, x). Using these tables, we can see how one function's changes to x are undone by the other function:

$$(f^{-1} \circ f)(1300) = f^{-1}(f(1300)) = f^{-1}(1000) = 1300.$$

The table for f shows that $f(1300) = 1000$.

The table for f^{-1} shows that $f^{-1}(1000) = 1300$.

The final price of the computer, \$1300, is identical to its starting price, \$1300.

With these ideas in mind, we present the formal definition of the inverse of a function:

Study Tip

The notation f^{-1} represents the inverse function of f. The -1 is *not* an exponent. The notation f^{-1} does *not* mean $\frac{1}{f}$:

$$f^{-1} \neq \frac{1}{f}.$$

Definition of the Inverse of a Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f.$$

The function g is the **inverse of the function f** and is denoted by f^{-1} (read "f-inverse"). Thus, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. The domain of f is equal to the range of f^{-1}, and vice versa.

1 Verify inverse functions.

EXAMPLE 1 Verifying Inverse Functions

Show that each function is the inverse of the other:

$$f(x) = 3x + 2 \quad \text{and} \quad g(x) = \frac{x-2}{3}.$$

Solution To show that f and g are inverses of each other, we must show that $f(g(x)) = x$ and $g(f(x)) = x$. We begin with $f(g(x))$.

$$f(x) = 3x + 2 \quad \text{This is the equation for } f.$$

Replace x with $g(x)$.

$$f(g(x)) = 3g(x) + 2 = 3\left(\frac{x-2}{3}\right) + 2 = x - 2 + 2 = x$$

$g(x) = \frac{x-2}{3}$

Next, we find $g(f(x))$.

$$g(x) = \frac{x-2}{3} \quad \text{This is the equation for } g.$$

Replace x with $f(x)$.

$$g(f(x)) = \frac{f(x) - 2}{3} = \frac{(3x+2) - 2}{3} = \frac{3x}{3} = x$$

$f(x) = 3x + 2$

Because g is the inverse of f (and vice versa), we can use inverse notation and write

$$f(x) = 3x + 2 \quad \text{and} \quad f^{-1}(x) = \frac{x-2}{3}.$$

Notice how f^{-1} undoes the changes produced by f: f changes x by *multiplying* by 3 and *adding* 2, and f^{-1} undoes this by *subtracting* 2 and *dividing* by 3. This "undoing" process is illustrated in **Figure 1.66**.

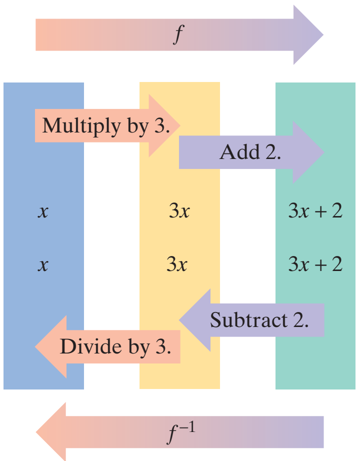


Figure 1.66 f^{-1} undoes the changes produced by f.

Check Point 1 Show that each function is the inverse of the other:

$$f(x) = 4x - 7 \quad \text{and} \quad g(x) = \frac{x+7}{4}.$$

2 Find the inverse of a function.

Finding the Inverse of a Function

The definition of the inverse of a function tells us that the domain of f is equal to the range of f^{-1}, and vice versa. This means that if the function f is the set of ordered pairs (x, y), then the inverse of f is the set of ordered pairs (y, x). If a function is defined by an equation, we can obtain the equation for f^{-1}, the inverse of f, by interchanging the role of x and y in the equation for the function f.

Finding the Inverse of a Function

The equation for the inverse of a function f can be found as follows:

1. Replace $f(x)$ with y in the equation for $f(x)$.
2. Interchange x and y.
3. Solve for y. If this equation does not define y as a function of x, the function f does not have an inverse function and this procedure ends. If this equation does define y as a function of x, the function f has an inverse function.
4. If f has an inverse function, replace y in step 3 by $f^{-1}(x)$. We can verify our result by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Study Tip

The procedure for finding a function's inverse uses a *switch-and-solve* strategy. Switch x and y, and then solve for y.

EXAMPLE 2 Finding the Inverse of a Function

Find the inverse of $f(x) = 7x - 5$.

Solution

Step 1 Replace $f(x)$ with y:

$$y = 7x - 5.$$

Step 2 Interchange x and y:

$$x = 7y - 5. \quad \text{This is the inverse function.}$$

Step 3 Solve for y:

$$x + 5 = 7y \quad \text{Add 5 to both sides.}$$

$$\frac{x+5}{7} = y. \quad \text{Divide both sides by 7.}$$

Step 4 Replace y with $f^{-1}(x)$:

$$f^{-1}(x) = \frac{x+5}{7}. \quad \text{The equation is written with } f^{-1} \text{ on the left.}$$

Thus, the inverse of $f(x) = 7x - 5$ is $f^{-1}(x) = \dfrac{x+5}{7}$.

The inverse function, f^{-1}, undoes the changes produced by f. f changes x by multiplying by 7 and subtracting 5. f^{-1} undoes this by adding 5 and dividing by 7.

Discovery

In Example 2, we found that if $f(x) = 7x - 5$, then

$$f^{-1}(x) = \frac{x+5}{7}.$$

Verify this result by showing that

$$f(f^{-1}(x)) = x$$

and

$$f^{-1}(f(x)) = x.$$

Check Point 2 Find the inverse of $f(x) = 2x + 7$.

EXAMPLE 3 Finding the Inverse of a Function

Find the inverse of $f(x) = x^3 + 1$.

Solution

Step 1 Replace $f(x)$ with y: $y = x^3 + 1$.

Step 2 Interchange x and y: $x = y^3 + 1$.

Step 3 Solve for y:

Our goal is to isolate y. Because $\sqrt[3]{y^3} = y$, we will take the cube root of both sides of the equation.

$$x - 1 = y^3 \quad \text{Subtract 1 from both sides.}$$

$$\sqrt[3]{x-1} = \sqrt[3]{y^3} \quad \text{Take the cube root on both sides.}$$

$$\sqrt[3]{x-1} = y. \quad \text{Simplify.}$$

Step 4 Replace y with $f^{-1}(x)$: $f^{-1}(x) = \sqrt[3]{x-1}$.

Thus, the inverse of $f(x) = x^3 + 1$ is $f^{-1}(x) = \sqrt[3]{x-1}$.

Check Point 3 Find the inverse of $f(x) = 4x^3 - 1$.

EXAMPLE 4 Finding the Inverse of a Function

Find the inverse of $f(x) = \frac{5}{x} + 4$.

Solution

Step 1 Replace $f(x)$ with y:

$$y = \frac{5}{x} + 4.$$

Step 2 Interchange x and y:

$$x = \frac{5}{y} + 4.$$

Our goal is to isolate y. To get y out of the denominator, we will multiply both sides of the equation by y, $y \neq 0$.

Step 3 Solve for y:

$$x = \frac{5}{y} + 4 \quad \text{This is the equation from step 2.}$$

$$xy = \left(\frac{5}{y} + 4\right)y \quad \text{Multiply both sides by } y,\ y \neq 0.$$

$$xy = \frac{5}{y} \cdot y + 4y \quad \text{Use the distributive property.}$$

$$xy = 5 + 4y \quad \text{Simplify: } \frac{5}{y} \cdot y = 5.$$

$$xy - 4y = 5 \quad \text{Subtract } 4y \text{ from both sides.}$$

$$y(x - 4) = 5 \quad \text{Factor out } y \text{ from } xy - 4y \text{ to obtain a single occurrence of } y.$$

$$\frac{y(x-4)}{x-4} = \frac{5}{x-4} \quad \text{Divide both sides by } x - 4,\ x \neq 4.$$

$$y = \frac{5}{x-4}. \quad \text{Simplify.}$$

Step 4 Replace y with $f^{-1}(x)$:

$$f^{-1}(x) = \frac{5}{x-4}.$$

Thus, the inverse of $f(x) = \frac{5}{x} + 4$ is $f^{-1}(x) = \frac{5}{x-4}$.

Check Point 4 Find the inverse of $f(x) = \frac{3}{x} - 1$.

---

3 Use the horizontal line test to determine if a function has an inverse function.

The Horizontal Line Test and One-to-One Functions

Let's see what happens if we try to find the inverse of the standard quadratic function, $f(x) = x^2$.

Step 1 Replace $f(x)$ with y: $y = x^2$.

Step 2 Interchange x and y: $x = y^2$.

Step 3 Solve for y: We apply the square root property to solve $y^2 = x$ for y. We obtain

$$y = \pm\sqrt{x}.$$

The $\pm$ in $y = \pm\sqrt{x}$ shows that for certain values of x (all positive real numbers), there are two values of y. Because this equation does not represent y as a function of x, the standard quadratic function $f(x) = x^2$ does not have an inverse function.

We can use a few of the solutions of $y = x^2$ to illustrate numerically that this function does not have an inverse:

Four solutions of $y = x^2$: $(-2, 4)$, $(-1, 1)$, $(1, 1)$, $(2, 4)$

Interchange x and y in each ordered pair.: $(4, -2)$, $(1, -1)$, $(1, 1)$, $(4, 2)$.

The input 1 is associated with two outputs, -1 and 1.

The input 4 is associated with two outputs, -2 and 2.

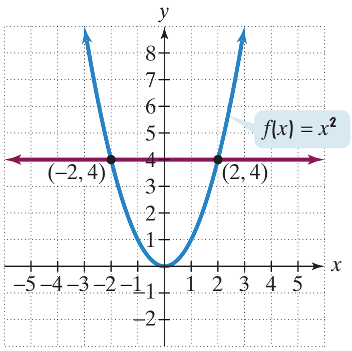


Figure 1.67 The horizontal line intersects the graph twice.

A function provides exactly one output for each input. Thus, the ordered pairs in the bottom row do not define a function.

Can we look at the graph of a function and tell if it represents a function with an inverse? Yes. The graph of the standard quadratic function $f(x) = x^2$ is shown in **Figure 1.67**. Four units above the x-axis, a horizontal line is drawn. This line intersects the graph at two of its points, $(-2, 4)$ and $(2, 4)$. Inverse functions have ordered pairs with the coordinates reversed. We just saw what happened when we interchanged x and y. We obtained $(4, -2)$ and $(4, 2)$, and these ordered pairs do not define a function.

If any horizontal line, such as the one in **Figure 1.67**, intersects a graph at two or more points, the set of these points will not define a function when their coordinates are reversed. This suggests the **horizontal line test** for inverse functions.

Discovery

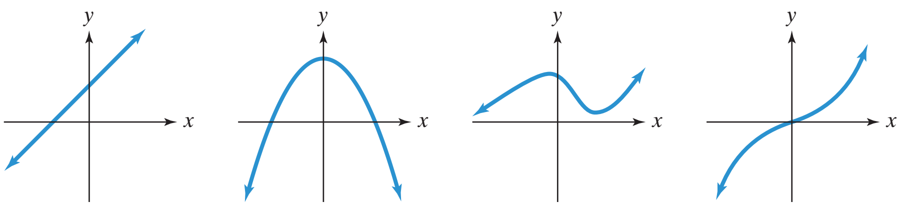
How might you restrict the domain of $f(x) = x^2$, graphed in **Figure 1.67**, so that the remaining portion of the graph passes the horizontal line test?

The Horizontal Line Test for Inverse Functions

A function f has an inverse that is a function, f^{-1}, if there is no horizontal line that intersects the graph of the function f at more than one point.

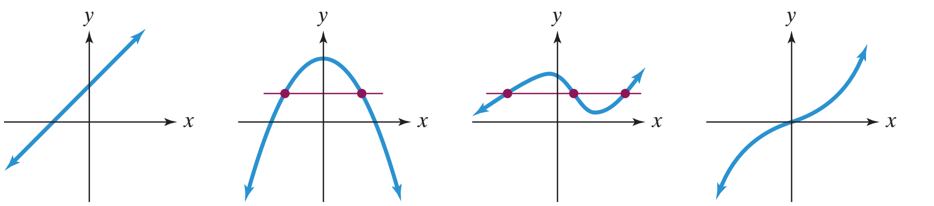
EXAMPLE 5 Applying the Horizontal Line Test

Which of the following graphs represent functions that have inverse functions?



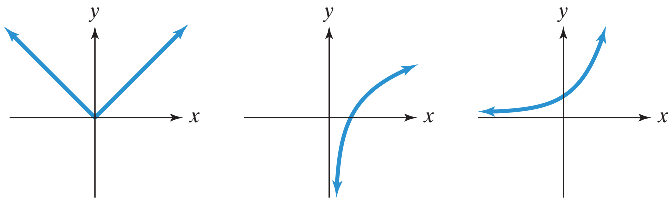
(a) (b) (c) (d)

Solution Notice that horizontal lines can be drawn in graphs **(b)** and **(c)** that intersect the graphs more than once. These graphs do not pass the horizontal line test. These are not the graphs of functions with inverse functions. By contrast, no horizontal line can be drawn in graphs **(a)** and **(d)** that intersects the graphs more than once. These graphs pass the horizontal line test. Thus, the graphs in parts **(a)** and **(d)** represent functions that have inverse functions.



Has an inverse function **(a)** | No inverse function **(b)** | No inverse function **(c)** | Has an inverse function **(d)**

Check Point 5 Which of the following graphs represent functions that have inverse functions?



A function passes the horizontal line test when no two different ordered pairs have the same second component. This means that if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. Such a function is called a **one-to-one function**. Thus, **a one-to-one function is a function in which no two different ordered pairs have the same second component. Only one-to-one functions have inverse functions.** Any function that passes the horizontal line test is a one-to-one function. Any one-to-one function has a graph that passes the horizontal line test.

4 Use the graph of a one-to-one function to graph its inverse function.

Graphs of f and f^{-1}

There is a relationship between the graph of a one-to-one function, f, and its inverse, f^{-1}. Because inverse functions have ordered pairs with the coordinates interchanged, if the point (a, b) is on the graph of f, then the point (b, a) is on the graph of f^{-1}. The points (a, b) and (b, a) are symmetric with respect to the line $y = x$. Thus, **the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.** This is illustrated in **Figure 1.68**.

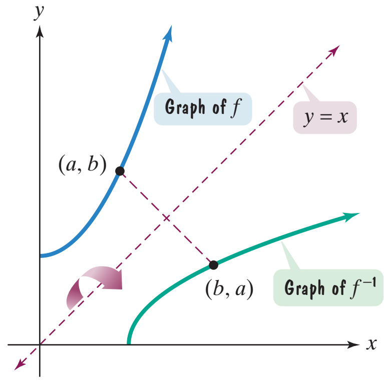


Figure 1.68 The graph of f^{-1} is a reflection of the graph of f about $y = x$.

EXAMPLE 6 Graphing the Inverse Function

Use the graph of f in **Figure 1.69** to draw the graph of its inverse function.

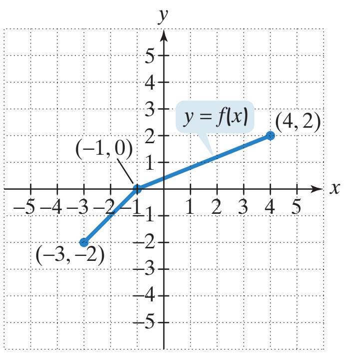


Figure 1.69

Solution We begin by noting that no horizontal line intersects the graph of f at more than one point, so f does have an inverse function. Because the points $(-3, -2)$, $(-1, 0)$, and $(4, 2)$ are on the graph of f, the graph of the inverse function, f^{-1}, has points with these ordered pairs reversed. Thus, $(-2, -3)$, $(0, -1)$, and $(2, 4)$ are on the graph of f^{-1}. We can use these points to graph f^{-1}. The graph of f^{-1} is shown in green in **Figure 1.70**. Note that the green graph of f^{-1} is the reflection of the blue graph of f about the line $y = x$.

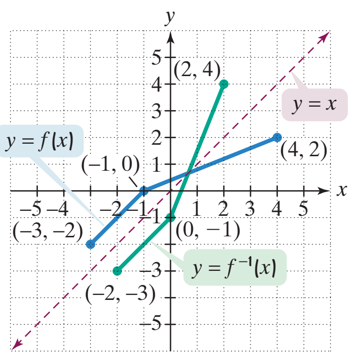


Figure 1.70 The graphs of f and f^{-1}

Check Point 6 The graph of function f consists of two line segments, one segment from $(-2, -2)$ to $(-1, 0)$ and a second segment from $(-1, 0)$ to $(1, 2)$. Graph f and use the graph to draw the graph of its inverse function.

5 Find the inverse of a function and graph both functions on the same axes.

In our final example, we will first find f^{-1}. Then we will graph f and f^{-1} in the same rectangular coordinate system.

EXAMPLE 7 Finding the Inverse of a Domain-Restricted Function

Find the inverse of $f(x) = x^2 - 1$ if $x \geq 0$. Graph f and f^{-1} in the same rectangular coordinate system.

Solution The graph of $f(x) = x^2 - 1$ is the graph of the standard quadratic function shifted vertically down 1 unit. **Figure 1.71** shows the function's graph. This graph fails the horizontal line test, so the function $f(x) = x^2 - 1$ does not have an inverse function. By restricting the domain to $x \geq 0$, as given, we obtain a new function whose graph is shown in red in **Figure 1.71**. This red portion of the graph is increasing on the interval $(0, \infty)$ and passes the horizontal line test. This tells us that $f(x) = x^2 - 1$ has an inverse function if we restrict its domain to $x \geq 0$. We use our four-step procedure to find this inverse function. Begin with $f(x) = x^2 - 1$, $x \geq 0$.

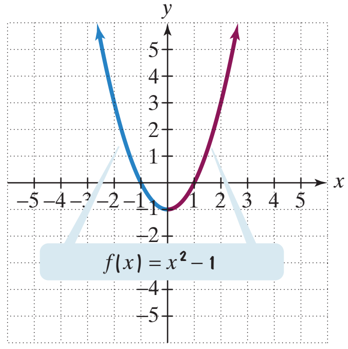


Figure 1.71

Step 1 Replace $f(x)$ with y: $y = x^2 - 1$, $x \geq 0$.

Step 2 Interchange x and y: $x = y^2 - 1$, $y \geq 0$.

Step 3 Solve for y:

$$x = y^2 - 1, y \geq 0$$ This is the equation from step 2.

$$x + 1 = y^2$$ Add 1 to both sides.

$$\sqrt{x+1} = y$$ Apply the square root property.

Because $y \geq 0$, take only the principal square root and not the negative square root.

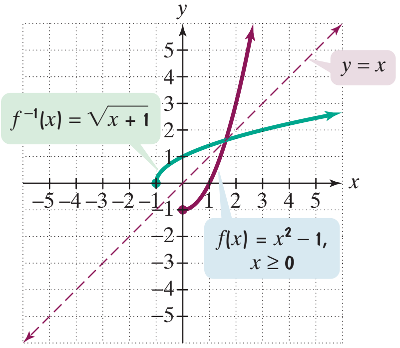


Figure 1.72

Step 4 Replace y with $f^{-1}(x)$: $f^{-1}(x) = \sqrt{x+1}$.

Thus, the inverse of $f(x) = x^2 - 1$, $x \geq 0$, is $f^{-1}(x) = \sqrt{x+1}$. The graphs of f and f^{-1} are shown in **Figure 1.72**. We obtained the graph of $f^{-1}(x) = \sqrt{x+1}$ by shifting the graph of the square root function, $y = \sqrt{x}$, horizontally to the left 1 unit. Note that the green graph of f^{-1} is the reflection of the red graph of f about the line $y = x$.

Check Point 7 Find the inverse of $f(x) = x^2 + 1$ if $x \geq 0$. Graph f and f^{-1} in the same rectangular coordinate system.

Exercise Set 1.8

Practice Exercises

In Exercises 1–10, find $f(g(x))$ and $g(f(x))$ and determine whether each pair of functions f and g are inverses of each other.

1. $f(x) = 4x$ and $g(x) = \dfrac{x}{4}$
2. $f(x) = 6x$ and $g(x) = \dfrac{x}{6}$
3. $f(x) = 3x + 8$ and $g(x) = \dfrac{x - 8}{3}$
4. $f(x) = 4x + 9$ and $g(x) = \dfrac{x - 9}{4}$
5. $f(x) = 5x - 9$ and $g(x) = \dfrac{x + 5}{9}$
6. $f(x) = 3x - 7$ and $g(x) = \dfrac{x + 3}{7}$
7. $f(x) = \dfrac{3}{x - 4}$ and $g(x) = \dfrac{3}{x} + 4$
8. $f(x) = \dfrac{2}{x - 5}$ and $g(x) = \dfrac{2}{x} + 5$
9. $f(x) = -x$ and $g(x) = -x$
10. $f(x) = \sqrt[3]{x - 4}$ and $g(x) = x^3 + 4$

The functions in Exercises 11–28 are all one-to-one. For each function,

a. *Find an equation for $f^{-1}(x)$, the inverse function.*

b. *Verify that your equation is correct by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.*

11. $f(x) = x + 3$

12. $f(x) = x + 5$

13. $f(x) = 2x$

14. $f(x) = 4x$

15. $f(x) = 2x + 3$

16. $f(x) = 3x - 1$

17. $f(x) = x^3 + 2$

18. $f(x) = x^3 - 1$

19. $f(x) = (x + 2)^3$

20. $f(x) = (x - 1)^3$

21. $f(x) = \dfrac{1}{x}$

22. $f(x) = \dfrac{2}{x}$

23. $f(x) = \sqrt{x}$

24. $f(x) = \sqrt[3]{x}$

25. $f(x) = \dfrac{7}{x} - 3$

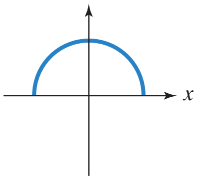
26. $f(x) = \dfrac{4}{x} + 9$

27. $f(x) = \dfrac{2x + 1}{x - 3}$

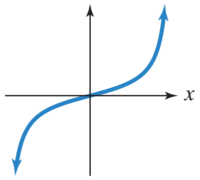
28. $f(x) = \dfrac{2x - 3}{x + 1}$

Which graphs in Exercises 29–34 represent functions that have inverse functions?

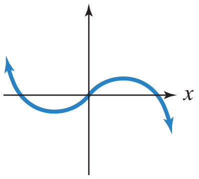
29.



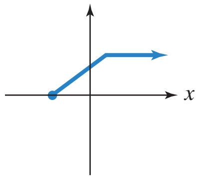
30.



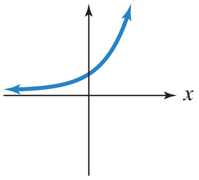
31.



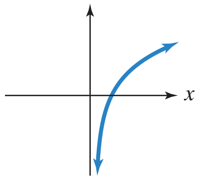
32.



33.

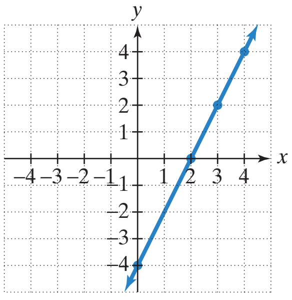


34.

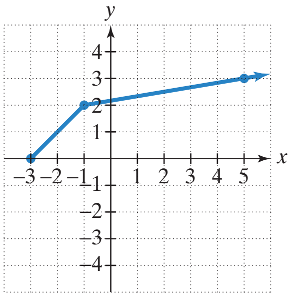


In Exercises 35–38, use the graph of f to draw the graph of its inverse function.

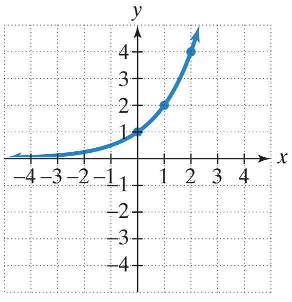
35.



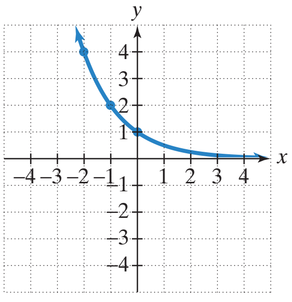
36.



37.



38.



In Exercises 39–52,

a. *Find an equation for $f^{-1}(x)$.*

b. *Graph f and f^{-1} in the same rectangular coordinate system.*

c. *Use interval notation to give the domain and the range of f and f^{-1}.*

39. $f(x) = 2x - 1$

40. $f(x) = 2x - 3$

41. $f(x) = x^2 - 4, x \geq 0$

42. $f(x) = x^2 - 1, x \leq 0$

43. $f(x) = (x - 1)^2, x \leq 1$

44. $f(x) = (x - 1)^2, x \geq 1$

45. $f(x) = x^3 - 1$

46. $f(x) = x^3 + 1$

47. $f(x) = (x + 2)^3$

48. $f(x) = (x - 2)^3$

(Hint for Exercises 49–52: To solve for a variable involving an nth root, raise both sides of the equation to the nth power: $(\sqrt[n]{y})^n = y$.)

49. $f(x) = \sqrt{x - 1}$

50. $f(x) = \sqrt{x} + 2$

51. $f(x) = \sqrt[3]{x} + 1$

52. $f(x) = \sqrt[3]{x - 1}$

Practice Plus

In Exercises 53–58, f and g are defined by the following tables. Use the tables to evaluate each composite function.

x	$f(x)$
−1	1
0	4
1	5
2	−1

x	$g(x)$
−1	0
1	1
4	2
10	−1

53. $f(g(1))$

54. $f(g(4))$

55. $(g \circ f)(-1)$

56. $(g \circ f)(0)$

57. $f^{-1}(g(10))$

58. $f^{-1}(g(1))$

In Exercises 59–64, let

$$f(x) = 2x - 5$$
$$g(x) = 4x - 1$$
$$h(x) = x^2 + x + 2.$$

Evaluate the indicated function without finding an equation for the function.

59. $(f \circ g)(0)$

60. $(g \circ f)(0)$

61. $f^{-1}(1)$

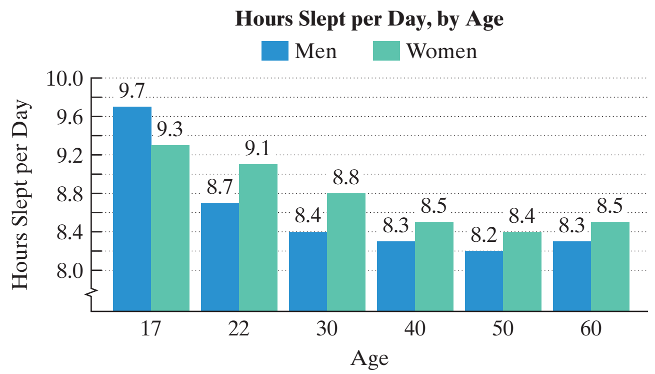
62. $g^{-1}(7)$

63. $g(f[h(1)])$

64. $f(g[h(1)])$

Application Exercises

The bar graph shows the average number of hours that Americans sleep per day, by age group. Use this information to solve Exercises 65–66.



Source: ATUS, Bureau of Labor Statistics

(In Exercises 65–66, refer to the graph at the bottom of the previous page.)

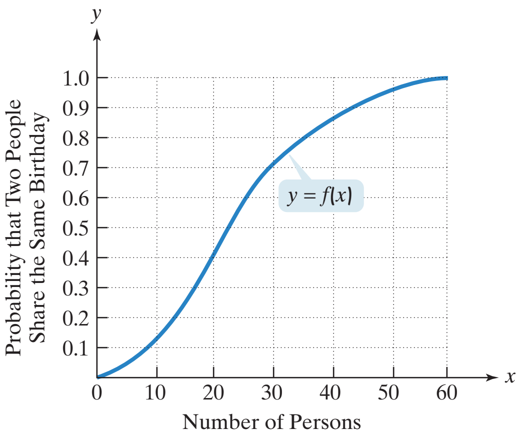
65. a. Consider a function, f, whose domain is the set of six ages shown. Let the range be the average number of hours that men sleep per day. Write the function f as a set of ordered pairs.

b. Write the relation that is the inverse of f as a set of ordered pairs. Based on these ordered pairs, is f a one-to-one function? Explain your answer.

66. a. Consider a function, g, whose domain is the set of six ages shown. Let the range be the average number of hours that women sleep per day. Write the function g as a set of ordered pairs.

b. Write the relation that is the inverse of g as a set of ordered pairs. Based on these ordered pairs, is g a one-to-one function? Explain your answer.

67. The graph represents the probability of two people in the same room sharing a birthday as a function of the number of people in the room. Call the function f.

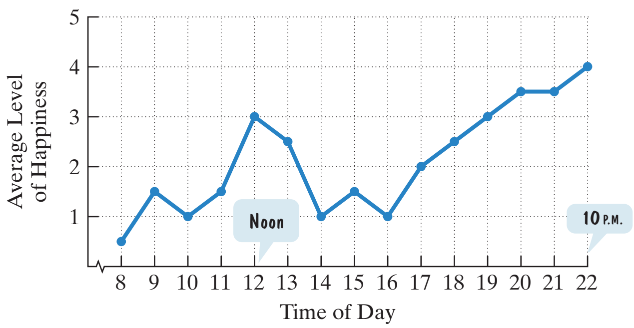


a. Explain why f has an inverse that is a function.

b. Describe in practical terms the meaning of $f^{-1}(0.25)$, $f^{-1}(0.5)$, and $f^{-1}(0.7)$.

68. A study of 900 working women in Texas showed that their feelings changed throughout the day. As the graph indicates, the women felt better as time passed, except for a blip (that's slang for relative maximum) at lunchtime.

Average Level of Happiness at Different Times of Day



Source: D. Kahneman et al., "A Survey Method for Characterizing Daily Life Experience," *Science*

a. Does the graph have an inverse that is a function? Explain your answer.

b. Identify two or more times of day when the average happiness level is 3. Express your answers as ordered pairs.

c. Do the ordered pairs in part (b) indicate that the graph represents a one-to-one function? Explain your answer.

69. The formula

$$y = f(x) = \frac{9}{5}x + 32$$

is used to convert from x degrees Celsius to y degrees Fahrenheit. The formula

$$y = g(x) = \frac{5}{9}(x - 32)$$

is used to convert from x degrees Fahrenheit to y degrees Celsius. Show that f and g are inverse functions.

Writing in Mathematics

70. Explain how to determine if two functions are inverses of each other.

71. Describe how to find the inverse of a one-to-one function.

72. What is the horizontal line test and what does it indicate?

73. Describe how to use the graph of a one-to-one function to draw the graph of its inverse function.

74. How can a graphing utility be used to visually determine if two functions are inverses of each other?

75. What explanations can you offer for the trends shown by the graph in Exercise 68?

Technology Exercises

In Exercises 76–83, use a graphing utility to graph the function. Use the graph to determine whether the function has an inverse that is a function (that is, whether the function is one-to-one).

76. $f(x) = x^2 - 1$

77. $f(x) = \sqrt[3]{2 - x}$

78. $f(x) = \frac{x^3}{2}$

79. $f(x) = \frac{x^4}{4}$

80. $f(x) = \text{int}(x - 2)$

81. $f(x) = |x - 2|$

82. $f(x) = (x - 1)^3$

83. $f(x) = -\sqrt{16 - x^2}$

In Exercises 84–86, use a graphing utility to graph f and g in the same viewing rectangle. In addition, graph the line y = x and visually determine if f and g are inverses.

84. $f(x) = 4x + 4$, $g(x) = 0.25x - 1$

85. $f(x) = \frac{1}{x} + 2$, $g(x) = \frac{1}{x - 2}$

86. $f(x) = \sqrt[3]{x} - 2$, $g(x) = (x + 2)^3$

Critical Thinking Exercises

Make Sense? *In Exercises 87–90, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

87. I found the inverse of $f(x) = 5x - 4$ in my head: The reverse of multiplying by 5 and subtracting 4 is adding 4 and dividing by 5, so $f^{-1}(x) = \frac{x + 4}{5}$.

88. I'm working with the linear function $f(x) = 3x + 5$ and I do not need to find f^{-1} in order to determine the value of $(f \circ f^{-1})(17)$.

Exercises 89–90 are based on the following cartoon.

B.C. by permission of Johnny Hart and Creators Syndicate, Inc.

89. Assuming that there is no such thing as metric crickets, I modeled the information in the first frame of the cartoon with the function

$$T(n) = \frac{n}{4} + 40,$$

where $T(n)$ is the temperature, in degrees Fahrenheit, and n is the number of cricket chirps per minute.

90. I used the function in Exercise 89 and found an equation for $T^{-1}(n)$, which expresses the number of cricket chirps per minute as a function of Fahrenheit temperature.

In Exercises 91–94, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

91. The inverse of $\{(1, 4), (2, 7)\}$ is $\{(2, 7), (1, 4)\}$.

92. The function $f(x) = 5$ is one-to-one.

93. If $f(x) = 3x$, then $f^{-1}(x) = \frac{1}{3x}$.

94. The domain of f is the same as the range of f^{-1}.

95. If $f(x) = 3x$ and $g(x) = x + 5$, find $(f \circ g)^{-1}(x)$ and $(g^{-1} \circ f^{-1})(x)$.

96. Show that

$$f(x) = \frac{3x - 2}{5x - 3}$$

is its own inverse.

97. *Freedom 7* was the spacecraft that carried the first American into space in 1961. Total flight time was 15 minutes and the spacecraft reached a maximum height of 116 miles. Consider a function, s, that expresses *Freedom 7*'s height, $s(t)$, in miles, after t minutes. Is s a one-to-one function? Explain your answer.

98. If $f(2) = 6$, and f is one-to-one, find x satisfying $8 + f^{-1}(x - 1) = 10$.

Group Exercise

99. In Tom Stoppard's play *Arcadia*, the characters dream and talk about mathematics, including ideas involving graphing, composite functions, symmetry, and lack of symmetry in things that are tangled, mysterious, and unpredictable. Group members should read the play. Present a report on the ideas discussed by the characters that are related to concepts that we studied in this chapter. Bring in a copy of the play and read appropriate excerpts.

Preview Exercises

Exercises 100–102 will help you prepare for the material covered in the next section.

100. Let $(x_1, y_1) = (7, 2)$ and $(x_2, y_2) = (1, -1)$. Find $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Express the answer in simplified radical form.

101. Use a rectangular coordinate system to graph the circle with center $(1, -1)$ and radius 1.

102. Solve by completing the square: $y^2 - 6y - 4 = 0$.

Section 1.9 Distance and Midpoint Formulas; Circles

Objectives

1. Find the distance between two points.
2. Find the midpoint of a line segment.
3. Write the standard form of a circle's equation.
4. Give the center and radius of a circle whose equation is in standard form.
5. Convert the general form of a circle's equation to standard form.

It's a good idea to know your way around a circle. Clocks, angles, maps, and compasses are based on circles. Circles occur everywhere in nature: in ripples on water, patterns on a moth's wings, and cross sections of trees. Some consider the circle to be the most pleasing of all shapes.

The rectangular coordinate system gives us a unique way of knowing a circle. It enables us to translate a circle's geometric definition into an algebraic equation. To do this, we must first develop a formula for the distance between any two points in rectangular coordinates.