

Polynomial and Rational Functions

2



We are surrounded by evidence that the world is profoundly mathematical. After turning a somersault, a diver's path can be modeled by a quadratic function, $f(x) = ax^2 + bx + c$, as can the path of a football tossed from quarterback to receiver or the path of a flipped coin. Even if you throw an object directly upward, although its path is straight and vertical, its changing height over time is described by a quadratic function. And tailgaters beware: whether you're driving a car, a motorcycle, or a truck on dry or wet roads, an array of quadratic functions that model your required stopping distances at various speeds are available to help you become a safer driver.

The quadratic functions surrounding our long history of throwing things appear throughout the chapter, including Example 5 in Section 2.2 and Example 4 in Section 2.7. Tailgaters should pay close attention to the Section 2.7 opener, Exercises 77–78 and 92–93 in Exercise Set 2.7, and Exercise 91 in the Chapter Review Exercises.

Section 2.1 Complex Numbers

Objectives

- 1 Add and subtract complex numbers.
- 2 Multiply complex numbers.
- 3 Divide complex numbers.
- 4 Perform operations with square roots of negative numbers.
- 5 Solve quadratic equations with complex imaginary solution.

Who is this kid warning us about our eyeballs turning black if we attempt to find the square root of -9 ? Don't believe what you hear on the street. Although square roots of negative numbers are not real numbers, they do play a significant role in algebra. In this section, we move beyond the real numbers and discuss square roots with negative radicands.

The Imaginary Unit i

In this section, we will study quadratic equations whose solutions involve the square roots of negative numbers. Because the square of a real number is never negative, there is no real number x such that $x^2 = -1$. To provide a setting in which such equations have solutions, mathematicians have invented an expanded system of numbers, the complex numbers. The *imaginary number* i , defined to be a solution of the equation $x^2 = -1$, is the basis of this new number system.



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The Imaginary Unit i

The **imaginary unit** i is defined as

$$i = \sqrt{-1}, \text{ where } i^2 = -1.$$

Using the imaginary unit i , we can express the square root of any negative number as a real multiple of i . For example,

$$\sqrt{-25} = \sqrt{-1} \sqrt{25} = i\sqrt{25} = 5i.$$

We can check this result by squaring $5i$ and obtaining -25 .

$$(5i)^2 = 5^2 i^2 = 25(-1) = -25$$

A new system of numbers, called *complex numbers*, is based on adding multiples of i , such as $5i$, to real numbers.

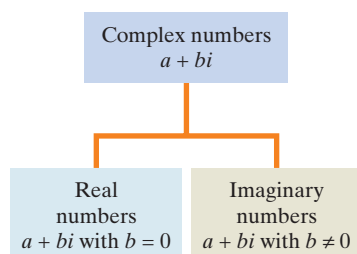


Figure 2.1 The complex number system

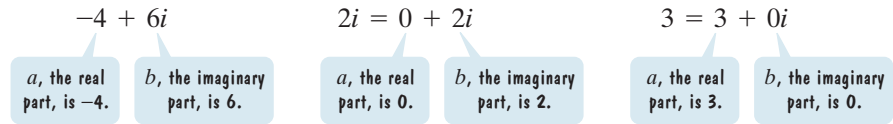
Complex Numbers and Imaginary Numbers

The set of all numbers in the form

$$a + bi,$$

with real numbers a and b , and i , the imaginary unit, is called the set of **complex numbers**. The real number a is called the **real part** and the real number b is called the **imaginary part** of the complex number $a + bi$. If $b \neq 0$, then the complex number is called an **imaginary number** (Figure 2.1). An imaginary number in the form bi is called a **pure imaginary number**.

Here are some examples of complex numbers. Each number can be written in the form $a + bi$.



Can you see that b , the imaginary part, is not zero in the first two complex numbers? Because $b \neq 0$, these complex numbers are imaginary numbers. Furthermore, the imaginary number $2i$ is a pure imaginary number. By contrast, the imaginary part of the complex number on the right is zero. This complex number is not an imaginary number. The number 3, or $3 + 0i$, is a real number.

A complex number is said to be **simplified** if it is expressed in the **standard form** $a + bi$. If b contains a radical, we usually write i before the radical. For example, we write $7 + 3i\sqrt{5}$ rather than $7 + 3\sqrt{5}i$, which could easily be confused with $7 + 3\sqrt{5}i$.

Expressed in standard form, two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Equality of Complex Numbers

$a + bi = c + di$ if and only if $a = c$ and $b = d$.

1 Add and subtract complex numbers.

Operations with Complex Numbers

The form of a complex number $a + bi$ is like the binomial $a + bx$. Consequently, we can add, subtract, and multiply complex numbers using the same methods we used for binomials, remembering that $i^2 = -1$.

Adding and Subtracting Complex Numbers

1. $(a + bi) + (c + di) = (a + c) + (b + d)i$

In words, this says that you add complex numbers by adding their real parts, adding their imaginary parts, and expressing the sum as a complex number.

2. $(a + bi) - (c + di) = (a - c) + (b - d)i$

In words, this says that you subtract complex numbers by subtracting their real parts, subtracting their imaginary parts, and expressing the difference as a complex number.

EXAMPLE 1 Adding and Subtracting Complex Numbers

Perform the indicated operations, writing the result in standard form:

a. $(5 - 11i) + (7 + 4i)$

b. $(-5 + i) - (-11 - 6i)$

Solution

a. $(5 - 11i) + (7 + 4i)$

$$= 5 - 11i + 7 + 4i$$

$$= 5 + 7 - 11i + 4i$$

$$= (5 + 7) + (-11 + 4)i$$

$$= 12 - 7i$$

Remove the parentheses.

Group real and imaginary terms.

Add real parts and add imaginary parts.

Simplify.

b. $(-5 + i) - (-11 - 6i)$

$$= -5 + i + 11 + 6i$$

$$= -5 + 11 + i + 6i$$

$$= (-5 + 11) + (1 + 6)i$$

$$= 6 + 7i$$

Remove the parentheses. Change signs of real and imaginary parts in the complex number being subtracted.

Group real and imaginary terms.

Add real parts and add imaginary parts.


Simplify.

Study Tip

The following examples, using the same integers as in Example 1, show how operations with complex numbers are just like operations with polynomials.

a. $(5 - 11x) + (7 + 4x)$
 $= 12 - 7x$

b. $(-5 + x) - (-11 - 6x)$
 $= -5 + x + 11 + 6x$
 $= 6 + 7x$

 **Check Point 1** Perform the indicated operations, writing the result in standard form:

a. $(5 - 2i) + (3 + 3i)$ b. $(2 + 6i) - (12 - i)$.

Multiplication of complex numbers is performed the same way as multiplication of polynomials, using the distributive property and the FOIL method. After completing the multiplication, we replace any occurrences of i^2 with -1 . This idea is illustrated in the next example.

2 Multiply complex numbers.

EXAMPLE 2 Multiplying Complex Numbers

Find the products:

a. $4i(3 - 5i)$ b. $(7 - 3i)(-2 - 5i)$.

Solution

a. $4i(3 - 5i)$

$$= 4i \cdot 3 - 4i \cdot 5i$$

Distribute $4i$ throughout the parentheses.

$$= 12i - 20i^2$$

Multiply.

$$= 12i - 20(-1)$$

Replace i^2 with -1 .

$$= 20 + 12i$$

Simplify to $12i + 20$ and write in standard form.

b. $(7 - 3i)(-2 - 5i)$

$$\begin{array}{cccc} \text{F} & \text{O} & \text{I} & \text{L} \\ = -14 - 35i + 6i + 15i^2 \end{array}$$

Use the FOIL method.

$$= -14 - 35i + 6i + 15(-1) \quad i^2 = -1$$

$$= -14 - 15 - 35i + 6i$$

Group real and imaginary terms.

$$= -29 - 29i$$

Combine real and imaginary terms.

 **Check Point 2** Find the products:

a. $7i(2 - 9i)$ b. $(5 + 4i)(6 - 7i)$.

3 Divide complex numbers.

Complex Conjugates and Division

It is possible to multiply imaginary numbers and obtain a real number. This occurs when we multiply $a + bi$ and $a - bi$.

$$\begin{array}{cccc} \text{F} & \text{O} & \text{I} & \text{L} \\ (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 & \text{Use the FOIL method.} \\ = a^2 - b^2(-1) & i^2 = -1 \\ = a^2 + b^2 & \text{Notice that this product eliminates } i. \end{array}$$

For the complex number $a + bi$, we define its *complex conjugate* to be $a - bi$. The multiplication of complex conjugates results in a real number.

Conjugate of a Complex Number

The **complex conjugate** of the number $a + bi$ is $a - bi$, and the complex conjugate of $a - bi$ is $a + bi$. The multiplication of complex conjugates gives a real number.

$$\begin{aligned} (a + bi)(a - bi) &= a^2 + b^2 \\ (a - bi)(a + bi) &= a^2 + b^2 \end{aligned}$$

Complex conjugates are used to divide complex numbers. The goal of the division procedure is to obtain a real number in the denominator. This real number becomes the denominator of a and b in the quotient $a + bi$. By multiplying the numerator and the denominator of the division by the complex conjugate of the denominator, you will obtain this real number in the denominator.

EXAMPLE 3 Using Complex Conjugates to Divide Complex Numbers

Divide and express the result in standard form: $\frac{7 + 4i}{2 - 5i}$.

Solution The complex conjugate of the denominator, $2 - 5i$, is $2 + 5i$. Multiplication of both the numerator and the denominator by $2 + 5i$ will eliminate i from the denominator while maintaining the value of the expression.

$$\frac{7 + 4i}{2 - 5i} = \frac{(7 + 4i)(2 + 5i)}{(2 - 5i)(2 + 5i)}$$

Multiply the numerator and the denominator by the complex conjugate of the denominator.

$$= \frac{\overset{\text{F}}{14} + \overset{\text{O}}{35i} + \overset{\text{I}}{8i} + \overset{\text{L}}{20i^2}}{2^2 + 5^2}$$

Use the FOIL method in the numerator and $(a - bi)(a + bi) = a^2 + b^2$ in the denominator.

$$= \frac{14 + 43i + 20(-1)}{29}$$

Combine imaginary terms and replace i^2 with -1 .

$$= \frac{-6 + 43i}{29}$$

Combine real terms in the numerator:
 $14 + 20(-1) = 14 - 20 = -6$.

$$= -\frac{6}{29} + \frac{43}{29}i$$

Express the answer in standard form.

Observe that the quotient is expressed in the standard form $a + bi$, with $a = -\frac{6}{29}$ and $b = \frac{43}{29}$.

Check Point 3 Divide and express the result in standard form: $\frac{5 + 4i}{4 - i}$.

- 4** Perform operations with square roots of negative numbers.

Roots of Negative Numbers

The square of $4i$ and the square of $-4i$ both result in -16 :

$$(4i)^2 = 16i^2 = 16(-1) = -16 \quad (-4i)^2 = 16i^2 = 16(-1) = -16.$$

Consequently, in the complex number system -16 has two square roots, namely, $4i$ and $-4i$. We call $4i$ the **principal square root** of -16 .

Principal Square Root of a Negative Number

For any positive real number b , the **principal square root** of the negative number $-b$ is defined by

$$\sqrt{-b} = i\sqrt{b}.$$

Consider the multiplication problem

$$5i \cdot 2i = 10i^2 = 10(-1) = -10.$$

This problem can also be given in terms of principal square roots of negative numbers:

$$\sqrt{-25} \cdot \sqrt{-4}.$$

Because the product rule for radicals only applies to real numbers, multiplying radicands is incorrect. **When performing operations with square roots of negative numbers, begin by expressing all square roots in terms of i .** Then perform the indicated operation.

Correct:

$$\begin{aligned}\sqrt{-25} \cdot \sqrt{-4} &= i\sqrt{25} \cdot i\sqrt{4} \\ &= 5i \cdot 2i \\ &= 10i^2 = 10(-1) = -10\end{aligned}$$

Incorrect:

~~$$\begin{aligned}\sqrt{-25} \cdot \sqrt{-4} &= \sqrt{(-25)(-4)} \\ &= \sqrt{100} \\ &= 10\end{aligned}$$~~

EXAMPLE 4 Operations Involving Square Roots of Negative Numbers

Perform the indicated operations and write the result in standard form:

a. $\sqrt{-18} - \sqrt{-8}$ b. $(-1 + \sqrt{-5})^2$ c. $\frac{-25 + \sqrt{-50}}{15}$.

Solution Begin by expressing all square roots of negative numbers in terms of i .

$$\begin{aligned}\text{a. } \sqrt{-18} - \sqrt{-8} &= i\sqrt{18} - i\sqrt{8} = i\sqrt{9 \cdot 2} - i\sqrt{4 \cdot 2} \\ &= 3i\sqrt{2} - 2i\sqrt{2} = i\sqrt{2}\end{aligned}$$

$$(A + B)^2 = A^2 + 2AB + B^2$$

$$\begin{aligned}\text{b. } (-1 + \sqrt{-5})^2 &= (-1 + i\sqrt{5})^2 = (-1)^2 + 2(-1)(i\sqrt{5}) + (i\sqrt{5})^2 \\ &= 1 - 2i\sqrt{5} + 5i^2 \\ &= 1 - 2i\sqrt{5} + 5(-1) \\ &= -4 - 2i\sqrt{5}\end{aligned}$$

$$\begin{aligned}\text{c. } \frac{-25 + \sqrt{-50}}{15} &= \frac{-25 + i\sqrt{50}}{15} \quad \sqrt{-b} = i\sqrt{b} \\ &= \frac{-25 + 5i\sqrt{2}}{15} \quad \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2} \\ &= \frac{-25}{15} + \frac{5i\sqrt{2}}{15} \quad \text{Write the complex number in standard form.} \\ &= -\frac{5}{3} + i\frac{\sqrt{2}}{3} \quad \text{Simplify.}\end{aligned}$$

Check Point 4 Perform the indicated operations and write the result in standard form:

a. $\sqrt{-27} + \sqrt{-48}$ b. $(-2 + \sqrt{-3})^2$ c. $\frac{-14 + \sqrt{-12}}{2}$.

- 5 Solve quadratic equations with complex imaginary solutions.

Study Tip

If you need to review quadratic equations and how to solve them, read Section P. 7, beginning on page 89.

Quadratic Equations with Complex Imaginary Solutions

We have seen that a quadratic equation can be expressed in the general form

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

All quadratic equations can be solved by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Recall that the quantity $b^2 - 4ac$, which appears under the radical sign in the quadratic formula, is called the discriminant. If the discriminant is negative, a quadratic equation has no real solutions. However, quadratic equations with negative discriminants do have two solutions. These solutions are imaginary numbers that are complex conjugates.

EXAMPLE 5 A Quadratic Equation with Imaginary Solutions

Solve using the quadratic formula: $3x^2 - 2x + 4 = 0$.

Solution The given equation is in general form. Begin by identifying the values for a , b , and c .

$$3x^2 - 2x + 4 = 0$$

$a = 3$
 $b = -2$
 $c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use the quadratic formula.

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(4)}}{2(3)}$$

Substitute the values for a , b , and c :
 $a = 3$, $b = -2$, and $c = 4$.

$$= \frac{2 \pm \sqrt{4 - 48}}{6}$$

$$-(-2) = 2 \text{ and } (-2)^2 = (-2)(-2) = 4.$$

$$= \frac{2 \pm \sqrt{-44}}{6}$$

Subtract under the radical. Because the number under the radical sign is negative, the solutions will not be real numbers.

$$= \frac{2 \pm 2i\sqrt{11}}{6}$$

$$\sqrt{-44} = \sqrt{4(11)(-1)} = 2i\sqrt{11}$$

$$= \frac{2(1 \pm i\sqrt{11})}{6}$$

Factor 2 from the numerator.

$$= \frac{1 \pm i\sqrt{11}}{3}$$

Divide numerator and denominator by 2.

$$= \frac{1}{3} \pm i\frac{\sqrt{11}}{3}$$

Write the complex numbers in standard form.

The solutions are complex conjugates, and the solution set is

$$\left\{ \frac{1}{3} + i\frac{\sqrt{11}}{3}, \frac{1}{3} - i\frac{\sqrt{11}}{3} \right\} \text{ or } \left\{ \frac{1}{3} \pm i\frac{\sqrt{11}}{3} \right\}.$$

Check Point 5 Solve using the quadratic formula:

$$x^2 - 2x + 2 = 0.$$

Exercise Set 2.1

Practice Exercises

In Exercises 1–8, add or subtract as indicated and write the result in standard form.

- $(7 + 2i) + (1 - 4i)$
- $(-2 + 6i) + (4 - i)$
- $(3 + 2i) - (5 - 7i)$
- $(-7 + 5i) - (-9 - 11i)$
- $6 - (-5 + 4i) - (-13 - i)$
- $7 - (-9 + 2i) - (-17 - i)$
- $8i - (14 - 9i)$
- $15i - (12 - 11i)$

In Exercises 9–20, find each product and write the result in standard form.

- $-3i(7i - 5)$
- $-8i(2i - 7)$
- $(-5 + 4i)(3 + i)$
- $(-4 - 8i)(3 + i)$
- $(7 - 5i)(-2 - 3i)$
- $(8 - 4i)(-3 + 9i)$
- $(3 + 5i)(3 - 5i)$
- $(2 + 7i)(2 - 7i)$
- $(-5 + i)(-5 - i)$
- $(-7 - i)(-7 + i)$
- $(2 + 3i)^2$
- $(5 - 2i)^2$

In Exercises 21–28, divide and express the result in standard form.

- $\frac{2}{3 - i}$
- $\frac{3}{4 + i}$
- $\frac{2i}{1 + i}$
- $\frac{5i}{2 - i}$
- $\frac{8i}{4 - 3i}$
- $\frac{-6i}{3 + 2i}$
- $\frac{2 + 3i}{2 + i}$
- $\frac{3 - 4i}{4 + 3i}$

In Exercises 29–44, perform the indicated operations and write the result in standard form.

- $\sqrt{-64} - \sqrt{-25}$
- $\sqrt{-81} - \sqrt{-144}$
- $5\sqrt{-16} + 3\sqrt{-81}$
- $5\sqrt{-8} + 3\sqrt{-18}$
- $(-2 + \sqrt{-4})^2$
- $(-5 - \sqrt{-9})^2$
- $(-3 - \sqrt{-7})^2$
- $(-2 + \sqrt{-11})^2$
- $\frac{-8 + \sqrt{-32}}{24}$
- $\frac{-12 + \sqrt{-28}}{32}$
- $\frac{-6 - \sqrt{-12}}{48}$
- $\frac{-15 - \sqrt{-18}}{33}$
- $\sqrt{-8}(\sqrt{-3} - \sqrt{5})$
- $\sqrt{-12}(\sqrt{-4} - \sqrt{2})$
- $(3\sqrt{-5})(-4\sqrt{-12})$
- $(3\sqrt{-7})(2\sqrt{-8})$

In Exercises 45–50, solve each quadratic equation using the quadratic formula. Express solutions in standard form.

- $x^2 - 6x + 10 = 0$
- $x^2 - 2x + 17 = 0$
- $4x^2 + 8x + 13 = 0$
- $2x^2 + 2x + 3 = 0$
- $3x^2 = 8x - 7$
- $3x^2 = 4x - 6$

Practice Plus

In Exercises 51–56, perform the indicated operation(s) and write the result in standard form.

- $(2 - 3i)(1 - i) - (3 - i)(3 + i)$
- $(8 + 9i)(2 - i) - (1 - i)(1 + i)$
- $(2 + i)^2 - (3 - i)^2$
- $(4 - i)^2 - (1 + 2i)^2$
- $5\sqrt{-16} + 3\sqrt{-81}$
- $5\sqrt{-8} + 3\sqrt{-18}$
- Evaluate $x^2 - 2x + 2$ for $x = 1 + i$.
- Evaluate $x^2 - 2x + 5$ for $x = 1 - 2i$.
- Evaluate $\frac{x^2 + 19}{2 - x}$ for $x = 3i$.
- Evaluate $\frac{x^2 + 11}{3 - x}$ for $x = 4i$.

Application Exercises

Complex numbers are used in electronics to describe the current in an electric circuit. Ohm's law relates the current in a circuit, I , in amperes, the voltage of the circuit, E , in volts, and the resistance of the circuit, R , in ohms, by the formula $E = IR$. Use this formula to solve Exercises 61–62.

- Find E , the voltage of a circuit, if $I = (4 - 5i)$ amperes and $R = (3 + 7i)$ ohms.
- Find E , the voltage of a circuit, if $I = (2 - 3i)$ amperes and $R = (3 + 5i)$ ohms.
- The mathematician Girolamo Cardano is credited with the first use (in 1545) of negative square roots in solving the now-famous problem, "Find two numbers whose sum is 10 and whose product is 40." Show that the complex numbers $5 + i\sqrt{15}$ and $5 - i\sqrt{15}$ satisfy the conditions of the problem. (Cardano did not use the symbolism $i\sqrt{15}$ or even $\sqrt{-15}$. He wrote R.m 15 for $\sqrt{-15}$, meaning "radix minus 15." He regarded the numbers $5 + \text{R.m } 15$ and $5 - \text{R.m } 15$ as "fictitious" or "ghost numbers," and considered the problem "manifestly impossible." But in a mathematically adventurous spirit, he exclaimed, "Nevertheless, we will operate.")

Writing in Mathematics

- What is i ?
- Explain how to add complex numbers. Provide an example with your explanation.

66. Explain how to multiply complex numbers and give an example.
67. What is the complex conjugate of $2 + 3i$? What happens when you multiply this complex number by its complex conjugate?
68. Explain how to divide complex numbers. Provide an example with your explanation.
69. Explain each of the three jokes in the cartoon on page 278.
70. A stand-up comedian uses algebra in some jokes, including one about a telephone recording that announces “You have just reached an imaginary number. Please multiply by i and dial again.” Explain the joke.

Explain the error in Exercises 71–72.

71. $\sqrt{-9} + \sqrt{-16} = \sqrt{-25} = i\sqrt{25} = 5i$

72. $(\sqrt{-9})^2 = \sqrt{-9} \cdot \sqrt{-9} = \sqrt{81} = 9$

Critical Thinking Exercises

Make Sense? In Exercises 73–76, determine whether each statement makes sense or does not make sense, and explain your reasoning.

73. The humor in this cartoon is based on the fact that “rational” and “real” have different meanings in mathematics and in everyday speech.



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74. The word *imaginary* in imaginary numbers tells me that these numbers are undefined.
75. By writing the imaginary number $5i$, I can immediately see that 5 is the constant and i is the variable.
76. When I add or subtract complex numbers, I am basically combining like terms.

In Exercises 77–80, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

77. Some irrational numbers are not complex numbers.

78. $(3 + 7i)(3 - 7i)$ is an imaginary number.

79. $\frac{7 + 3i}{5 + 3i} = \frac{7}{5}$

80. In the complex number system, $x^2 + y^2$ (the sum of two squares) can be factored as $(x + yi)(x - yi)$.

In Exercises 81–83, perform the indicated operations and write the result in standard form.

81. $\frac{4}{(2 + i)(3 - i)}$

82. $\frac{1 + i}{1 + 2i} + \frac{1 - i}{1 - 2i}$

83. $\frac{8}{1 + \frac{2}{i}}$

Preview Exercises

Exercises 84–86 will help you prepare for the material covered in the next section.

In Exercises 84–85, solve each quadratic equation by the method of your choice.

84. $0 = -2(x - 3)^2 + 8$

85. $-x^2 - 2x + 1 = 0$

86. Use the graph of $f(x) = x^2$ to graph $g(x) = (x + 3)^2 + 1$.