Benchmark Codes: MA.912.G.8.1

Lesson 11: Euclidean Geometry

Euclid's *Elements* contains a number of innovative mathematical definitions and postulates. In this lesson, you will review the meaning of these terms and distinguish between postulates, theorems, and undefined terms. You will apply all of these definitions, postulates, theorems, and undefined terms in proofs. You will create deductive proofs and use proofs to prove that a quadrilateral is a parallelogram, rhombus, trapezoid, kite, and/or isosceles trapezoid.

Definitions and Postulates

Euclid's Definitions

A **definition** is a statement that describes the meaning of a word. One reason *Elements* was so revolutionary is because it helped standardize the meaning of many mathematical terms.

Over the 13 books that comprise *Elements,* Euclid provided more than 100 definitions. The first four books cover two-dimensional geometry, and his definitions include essential math terms. His last three books cover three-dimensional geometry, and his definitions include the basics of solid figures, including such basic terms as *solid* ("that which has length, breadth, and depth") and *cube* ("a solid figure contained by six equal squares").

Euclid's Postulates

A **postulate** is a statement that is accepted as true. A postulate does not require proof because it is assumed to be true. Another word for a **postulate** is an **axiom**.

Euclid presented five significant postulates about planar geometry in *Elements*, as listed below.

Postulate 1

Euclid said a point was a location with "no part," meaning it has no width, length, or height. It is simply a position in space. When two points are connected, a line segment is formed with the points at its ends. According to Euclid's first postulate, a line segment can be drawn to connect any two points.

Postulate 2

A line segment has points at both of its ends, so it has a finite length. According to Euclid's second postulate, any line segment can be extended to create a line (which has an infinite length). This knowledge can be extremely valuable when solving geometry problems.

Postulate 3

Euclid's third postulate states that any line segment can be used to create a circle with the segment as its radius. You can do this on a piece of paper with a compass. This is an essential tool in measuring distances and creating equal line segments, since the distance from the center of a circle to any point on the circle is the same.

Postulate 4

In Euclid's fourth postulate, he states that all right angles are congruent. Every right angle has a measure of 90°. An angle with a measure of 90° is congruent to any another angle with a measure of 90°. Therefore, every right angle must be congruent to every other right angle.

Postulate 5

Euclid's fifth postulate state that if there is a line and a point not on the line, then there is exactly one line that goes through the point and is parallel to the line. It is important to note that this postulate only applies to planar geometry; it does not work with curves.

Benchmark Codes: MA.912.G.8.1

Practice

Directions: For questions 1 through 4, define the terms from Euclid's *Elements*.

1. cube

____

2. acute angle

____

3. diameter of a circle

____

4. equilateral triangle

____

Directions: For questions 5 through 9, fill in the blanks to complete one of Euclid's five postulates.

5. All right angles are always ______________.

6. A line segment can be used as the ______________ of a circle.

7. If points *A* and *B* lie in a plane, a ______________ can be used to connect the points.

8. A line segment can be extended beyond its endpoints to form a ______________.

9. Points *W* and *Z* lie on the same plane. Point *W* lies on line *a*. Point *Z* does not lie on line *a*. There is/are ______________ possible line(s) that contains point *Z* that is parallel to line *a*.

Constructive Proofs in Euclidean Geometry

In addition to the definitions and the postulates, Euclid's *Elements* included more than 400 important mathematical propositions. A **proposition** is a statement that must be either true or false. A proposition can be used to construct a theorem. A **theorem** is a statement that can be proven to be true.

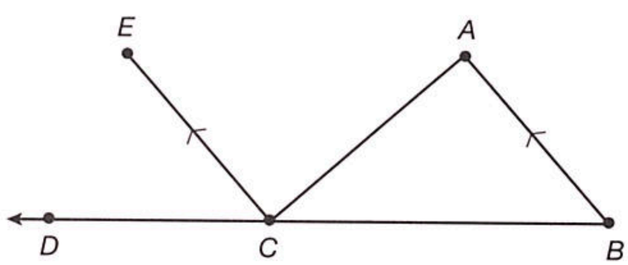
Euclid proved his propositions and theorems using definitions, postulates, or undefined terms. An **undefined term** is a term that should be clearly understood without additional explanation. No formal definition is necessary in a proof including these terms. Undefined terms in geometry are point, line, and plane.

Euclid used constructive proofs to prove many of his propositions and theorems. A constructive proof is a type of direct proof. It shows a statement to be true by showing how to create an object. For example, the very first proposition in the first book of *Elements* uses a constructive proof. In that proposition, Euclid constructs an equilateral triangle as proof.

Example

Use a constructive proof to show that the angle sum of a triangle is 180°.

Construct a triangle *ABC*, with an exterior angle $\angle DCA$. Then draw $\overline{EC}$ parallel to $\overline{AB}$.



Because $\overline{EC}$ is parallel to $\overline{AB}$, $\angle ACE \cong \angle CAB$ and $\angle ABC \cong \angle DCE$. It is clear that the measures of $\angle ACB$, $\angle ACE$ and $\angle DCE$ add to 180°, since they are on a straight line. Using substitution we find that:

$$180° = m\angle ACB + m\angle ACE + m\angle DCE$$

$$180° = m\angle ACB + m\angle CAB + m\angle ABC$$

This shows that the angle sum of a triangle is 180°. Note that you can construct the points *D* and *E* using a compass and straightedge given any triangle *ABC*. So this result is true for any triangle.

Benchmark Codes: MA.912.G.8.1

Practice

Directions: For questions 1 through 4, prove one of Euclid's propositions using a constructive proof and previous theorems.

1. Straight lines parallel to a straight line are also parallel to one another.

2. In any triangle the sum of the lengths of any two sides is greater than the length of the third side.

3. Two triangles have two sets of congruent sides, but their third sides are not equal. The included angle of the triangle opposite the larger third side will be larger than the included angle of the other triangle.

4. The sum of the opposite angles of a quadrilateral inscribed in a circle equals two right angles.

Benchmark Codes: MA.912.D.6.4, MA.912.G.8.1, MA.912.G.8.4, MA.912.G.8.5

Deductive Proofs in Euclidean Geometry

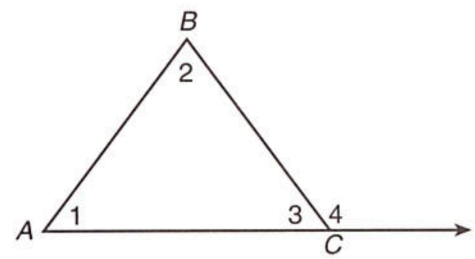
While Euclid frequently used constructive proofs in *Elements,* deductive reasoning is also used in Euclidean geometry. As mentioned on page 202, deductive reasoning is the process of arriving at a conclusion from premises that are given and accepted. A **deductive proof** uses flow proofs, paragraphs, two-column proofs, or indirect proofs to demonstrate deductive reasoning.

The following example demonstrates a two-column deductive proof.

Example

Given: $\angle 3$ is supplementary to $\angle 4$

Prove: $m\angle 1 + m\angle 2 = m\angle 4$



Proof:

Statements	Reasons
1. $\angle 3$ is supplementary to $\angle 4$	1. Given
2. $m\angle 3 + m\angle 4 = 180°$	2. Definition of supplementary angles
3. $m\angle 1 + m\angle 2 + m\angle 3 = 180°$	3. The sum of the measures of the interior angles of a triangle is 180°.
4. $m\angle 1 + m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$	4. Substitution property
5. $m\angle 1 + m\angle 2 = m\angle 4$	5. Subtraction property

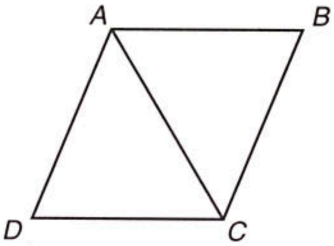
This proves that the measure of an exterior angle of a triangle equals the sum of the measures of the two remote interior angles.

Practice

Directions: For questions 1 and 2, complete the proofs.

1. **Given:** $\overline{DC} \parallel \overline{BA}$ and $\overline{DA} \parallel \overline{BC}$

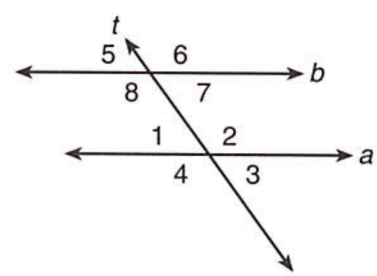
   Prove: $\triangle DAC \cong \triangle BCA$



Statements	Reasons
1. $\overline{DC} \parallel \overline{BA}$ and $\overline{DA} \parallel \overline{BC}$	1. Given
2. $\angle CAD \cong \angle ACB$	2. ______
3. $\angle ACD \cong \angle CAB$	3. ______
4. $\overline{AC} \cong \overline{CA}$	4. ______
5. $\triangle DAC \cong \triangle BCA$	5. ______

2. **Given:** $a \parallel b$

   Prove: $m\angle 2 + m\angle 5 = 180°$



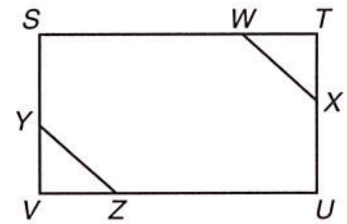
Statements	Reasons
1. $a \parallel b$	1. Given
2. $m\angle 2 + m\angle 1 = 180°$	2. ______
3. $\angle 1 \cong \angle 5$	3. ______
4. $m\angle 1 = m\angle 5$	4. ______
5. $m\angle 2 + m\angle 5 = 180°$	5. ______

Benchmark Codes: MA.912.D.6.4, MA.912.G.8.1, MA.912.G.8.4, MA.912.G.8.5

Directions: For questions 3 and 4, complete the proofs. Successful proofs might not use all of the lines given.

3. **Given:** $VZ = TW$, $VU = 3VZ$, and $TS = 3TW$

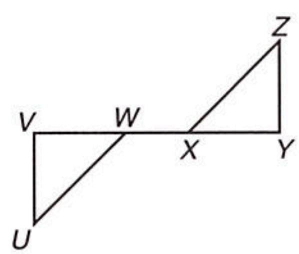
   Prove: $VU = TS$



Statements	**Reasons**
1. $VZ = TW$, $VU = 3VZ$, and $TS = 3TW$	1. Given
2. ______	2. ______
3. ______	3. ______
4. ______	4. ______
5. ______	5. ______
6. ______	6. ______
7. ______	7. ______

4. **Given:** $\overline{UW} \parallel \overline{ZX}$, $\overline{VX} \cong \overline{YW}$, and $\angle UVW \cong \angle ZYX$

   Prove: $\triangle UVW \cong \triangle ZYX$



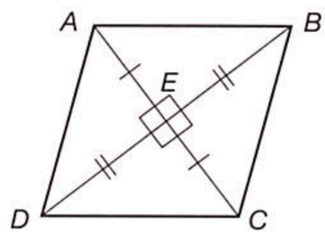
Statements	**Reasons**
1. $\overline{UW} \parallel \overline{ZX}$, $\overline{VX} \cong \overline{YW}$, and $\angle UVW \cong \angle ZYX$	1. Given
2. ______	2. ______
3. ______	3. ______
4. ______	4. ______
5. ______	5. ______
6. ______	6. ______
7. ______	7. ______
8. ______	8. ______
9. ______	9. ______

Quadrilaterals in Euclidean Geometry

The second book of Euclid's *Elements* focuses on algebra in geometry and includes two definitions, both about parallelograms. Several of the propositions provide proof that, given certain properties, a quadrilateral is a parallelogram, rhombus, trapezoid, kite, and/or an isosceles trapezoid.

Example

In quadrilateral *ABCD,* the diagonals bisect each other and are perpendicular. Use a proof to show that *ABCD* is a rhombus.



To prove that the figure is a rhombus we use the definition of a rhombus: a parallelogram with four congruent sides. This means we must prove that the two pairs of opposite sides are parallel and that the four sides are congruent. The proof follows on the next page.

Benchmark Codes: MA.912.G.3.4

Statement	**Reason**
1. $m\angle AEB = 90°$ $m\angle BEC = 90°$ $m\angle AED = 90°$ $m\angle BEC = 90°$	1. Given
2. $m\angle AEB = m\angle BEC = m\angle AED = m\angle BEC$	2. Substitution property
3. $AE = EC$	3. Given
4. $BE = ED$	4. Given
5. $\triangle AEB \cong \triangle CEB \cong \triangle AED \cong \triangle CED$	5. SAS congruence theorem
6. $AB = BC = CD = DA$	6. Def. of congruent triangles
7. $\angle ECD \cong \angle EAB$	7. Def. of congruent triangles
8. $\overline{AB} \parallel \overline{DC}$	8. Alt. interior angles are congruent.
9. $\angle ADE \cong \angle CBE$	9. Def. of congruent triangles
10. $\overline{AD} \parallel \overline{BC}$	10. Alt. interior angles are congruent.
11. *ABCD* is a rhombus	11. Def. of a rhombus

Since the four triangles share two common sides and have a common interior angle measure, all four triangles are congruent (Statement 5). Because the triangles are congruent, the four sides of the quadrilateral have the same measure (Statement 6). To show that the two pairs of sides are parallel we use the converse of a known fact: given two lines and a transversal, if the alternate interior angles are congruent, then the two lines are parallel (Statements 7 through 10). These statements have shown that the quadrilateral fulfills the definition of a rhombus: a parallelogram with four congruent sides.

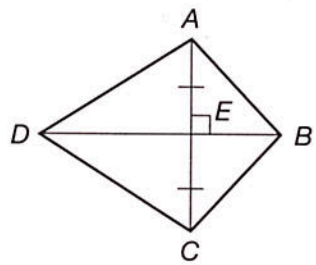
TIP: To conclude a proof you may write Q.E.D., which is Latin for *quod erat demonstrandum*: "that which was to be proven."

Practice

Directions: For questions 1 through 4, use a formal or informal proof to prove the given statement.

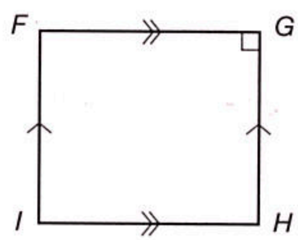
1. **Given:** The diagonals of *ABCD* intersect at point *E* to create right angle *AEB*. Diagonal $\overline{DB}$ bisects diagonal $\overline{AC}$.

   Prove: *ABCD* is a kite.



2. **Given:** The sides of *FGHI* are parallel and $m\angle FGH$ is 90°.

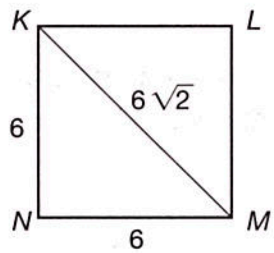
   Prove: *FGHI* is a rectangle.



Benchmark Codes: MA.912.G.3.4

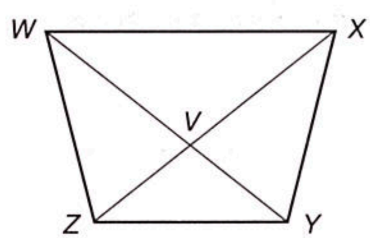
3. **Given:** $\triangle KLM \cong \triangle MNK$. $KN = 6$, $NM = 6$, and diagonal $KM = 6\sqrt{2}$.

   Prove: *KLMN* is a square.



4. **Given:** $WV = XV$, $VY = VZ$, and $m\angle XYZ > 90°$.

   Prove: *WXYZ* is an isosceles trapezoid.



Geometry Practice begins on the following page.