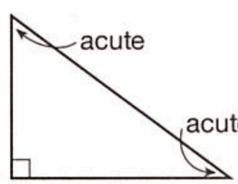
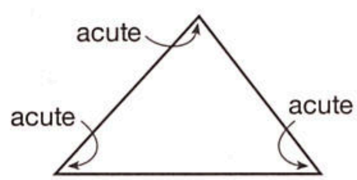
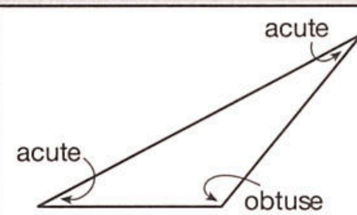
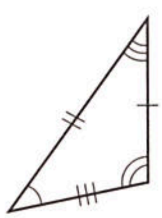
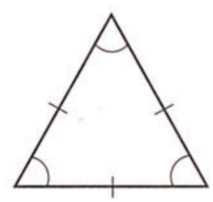
Benchmark Codes: MA.912.G.4.1

Lesson 3: Triangles

In this lesson, you will learn about the different types of triangles. You will use the triangle inequality and the hinge theorem. You will prove that triangles are congruent or similar and answer questions based on corresponding parts of congruent or similar triangles. You will construct congruent triangles. Finally, you will learn about the different segments of a triangle.

Types of Triangles

A **triangle** is a polygon that has three sides.

Triangles		
 **Right Triangle** one right angle	 **Acute Triangle** three acute angles	 **Obtuse Triangle** one obtuse angle
 **Scalene Triangle** no congruent sides and angles	 **Isosceles Triangle** at least two congruent sides and angles	 **Equilateral Triangle** three congruent sides and angles

Each side of a triangle is proportional to its opposite angle. The largest side of a triangle is opposite its largest angle. The smallest side of a triangle is opposite its smallest angle. If the lengths of two sides of a triangle are equal, then their opposite angles are congruent. In a similar way, if two angles of a triangle are congruent, then their opposite sides are equal in length.

Practice

Directions: Use the characteristics of triangles to answer questions 1 through 5. Name each triangle based on both its side lengths and its angle measures, if possible.

1. What type of triangle has three angles of equal measure?

____

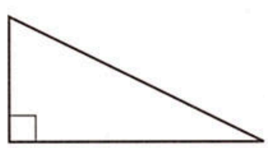
2. What type of triangle has exactly two sides of equal length?

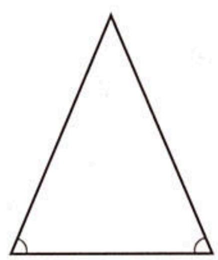
____

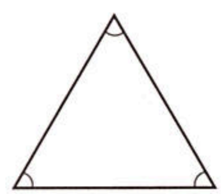
3. What type of triangle has an angle that measures 90°?

____

4. Classify each of the following triangles.







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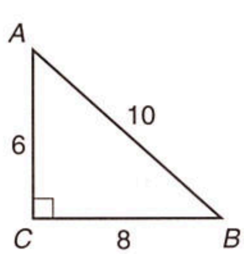
5. Order the angles of the triangle from the largest measure, to the smallest measure.



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Benchmark Codes: MA.912.G.4.7

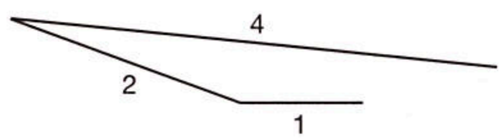
The Triangle Inequality and the Hinge Theorem

The **triangle inequality** states that the sum of the lengths of any two sides of a triangle must be greater than the length of the third side.

Example

A triangle has two side lengths of 1 and 2. Can the third side length of the triangle be 4?

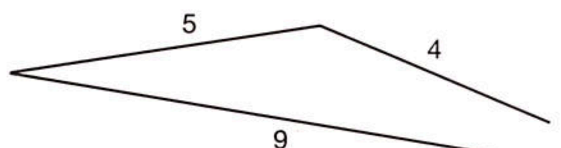
The sum of the two given side lengths is $1 + 2 = 3$. The triangle inequality says the sum must be greater than the third side length, so the third side cannot be 4. The following figure shows three line segments measuring 1, 2, and 4. The segments cannot be used to form a triangle.



Example

A triangle has two side lengths of 4 and 5. What is the range of values for the third side, *x*?

According to the triangle inequality, the sum of any two lengths of a triangle must be greater than the third side. The sum of the two given side lengths is $4 + 5 = 9$. The sum must be greater than the third side, so the third side must be less than 9. The following figure shows three line segments measuring 4, 5, and 9. The segments cannot be used to form a triangle.

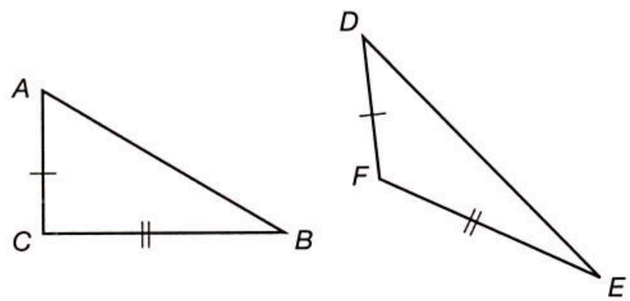


For the third line segment to create a triangle, its side length must be smaller than 9. The length of the line segment also must be greater than 1; otherwise the sum of the lengths of the two smaller line segments will not be greater than the longest side, 5. The possible range of values for the third side can be represented by the inequality $1 < x < 9$.

The **hinge theorem** states that if two sides of a triangle are congruent to two sides of another triangle, the triangle with the larger included angle will have a third side with a longer length. The converse is also true: If two sides of a triangle are congruent to two sides of another triangle, the triangle with the longer third side will have a larger included angle between the two given sides.

Example

In triangles *ABC* and *DEF*, $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$, and $m\angle ACB < m\angle DFE$. Is *AB* smaller than, equal to, or greater than *DE*?



Two sides of triangles *ABC* and *DEF* are congruent. According to the hinge theorem, the triangle with the larger included angle will have a third side with a longer length. It is also given that $m\angle ACB < m\angle DFE$, which means that $\triangle DEF$ has the greater included angle. The third side of $\triangle DEF$ will be greater than the third side of $\triangle ABC$.

Therefore, $AB < DE$.

Benchmark Codes: MA.912.G.4.7

Practice

Directions: For questions 1 through 6, write the range of possible values for the third side, *x*, of a triangle given its three side lengths.

1. 5, 5, x

   ____

2. 1, 9, x

   ____

3. 12, 4, x

   ____

4. 3, x, 10

   ____

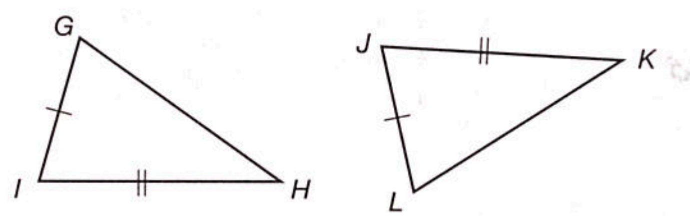
5. 2, x, 3

   ____

6. x, 12, 13

   ____

Directions: Answer questions 7 and 8 based on the following figures.



Given: $\overline{GI} \cong \overline{JL}$ and $\overline{IH} \cong \overline{JK}$

7. If $m\angle\overline{GIH} < m\angle\overline{LJK}$ and GH is 10, which is a possible side length for LK?

   A. 8
   B. 9
   C. 10
   D. 11

8. If $m\angle\overline{LJK} < m\angle\overline{GIH}$ and GH is 25, which is a possible side length for LK?

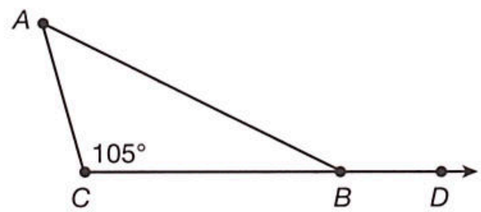
   A. 20
   B. 25
   C. 26
   D. 40

Exterior Angle Inequality

The exterior angle inequality states that the measure of an exterior angle of a triangle must be greater than the measure of either of the nonadjacent interior angles.

Example

In $\triangle ABC$, $m\angle BCA > m\angle CAB$. If the measure of $\angle ABD$ is an integer, what is the smallest possible value of $m\angle ABD$?

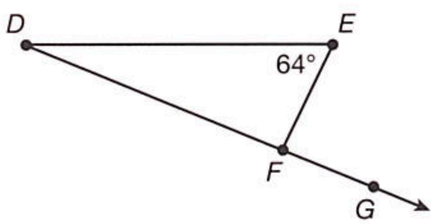


The measure of the exterior angle $\angle ABD$ must be greater than the measure of either of the nonadjacent interior angles. The measure of $\angle BCA$ is 105°. Therefore, $m\angle ABD$ must be greater than 105°. Because it is an integer, the smallest possible value of $m\angle ABD$ is 106°.

Practice

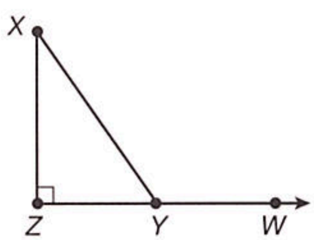
Directions: For questions 1 through 4, find the measure that the exterior angle must be greater than.

1.



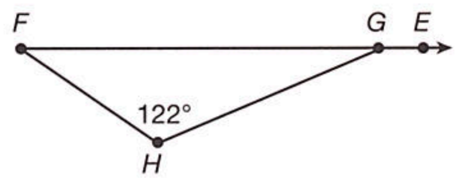
$m\angle EFG >$ ____________

2.



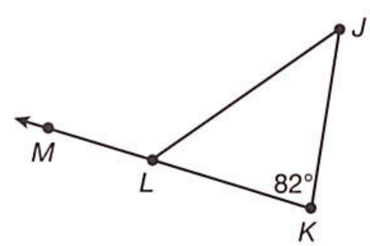
$m\angle XYW >$ ____________

3.



$m\angle HGE >$ ____________

4.



$m\angle JLM >$ ____________

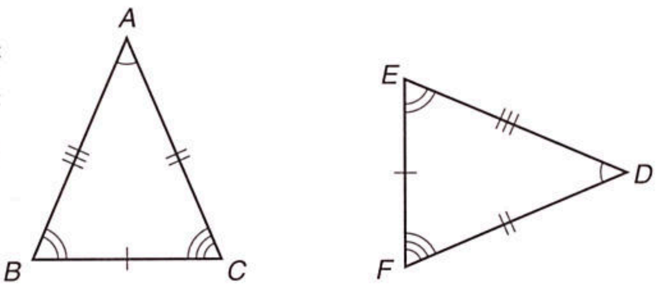
Benchmark Codes: MA.912.G.4.4, MA.912.G.4.6

Congruent Triangles

Congruent triangles have corresponding angles and corresponding sides that are congruent. The symbol for congruent is ≅.

Given: $\triangle ABC \cong \triangle DEF$

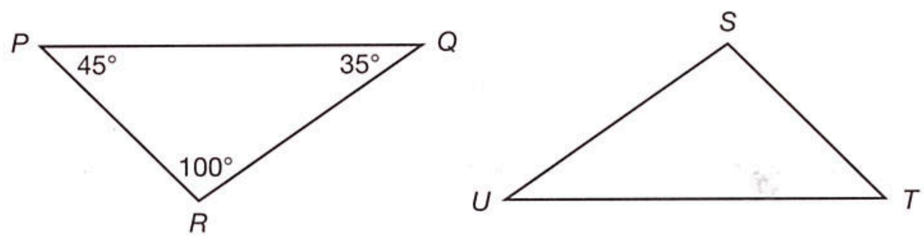
$\overline{AB} \cong \overline{DE}$
$\overline{AC} \cong \overline{DF}$
$\overline{BC} \cong \overline{EF}$



$\angle A \cong \angle D$
$\angle B \cong \angle E$
$\angle C \cong \angle F$

Example

Given $\triangle PQR \cong \triangle TUS$ what is the measure of $\angle SUT$?



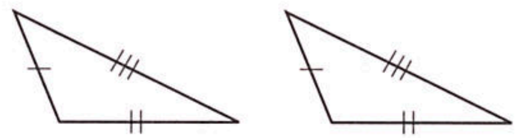
Because $\triangle PQR$ is congruent to $\triangle TUS$, they will have congruent corresponding angle measures. To determine the measure of $\angle SUT$, you must determine which angle it corresponds to in $\triangle PQR$.

The obtuse angles in the congruent triangles are $\angle R$ and $\angle S$. Therefore, $\triangle TUS$ is formed by rotating $\triangle PQR$ 180°. As a result, $\angle RPQ$ corresponds to $\angle STU$. Because the measure of $\angle RQP$ is 35°, the measure of $\angle SUT$ must also be 35°.

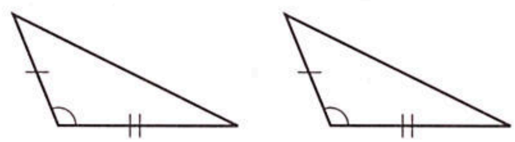
Proving Triangles Congruent Using Postulates

It is not necessary to compare all six pairs of corresponding parts of two triangles to prove they are congruent. A **postulate** (rule) can be used to prove two triangles are congruent by comparing only three pairs of corresponding parts.

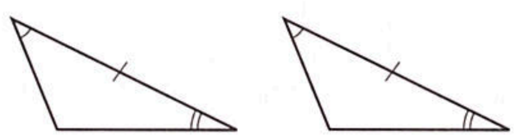
- **Side-Side-Side Congruence Postulate (SSS):** Two triangles are congruent if the three sides of one triangle are congruent to the corresponding three sides of the other triangle.



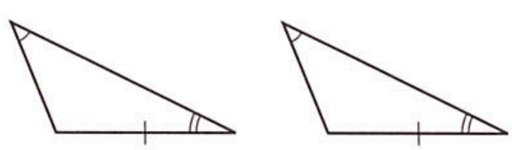
- **Side-Angle-Side Congruence Postulate (SAS):** Two triangles are congruent if two sides and the included angle of one triangle are congruent to the corresponding two sides and included angle of the other triangle.



- **Angle-Side-Angle Congruence Postulate (ASA):** Two triangles are congruent if two angles and the included side of one triangle are congruent to the corresponding two angles and included side of the other triangle.



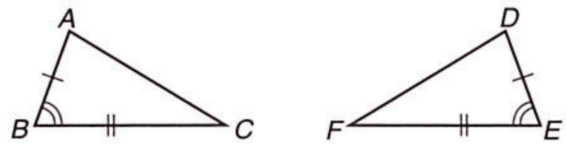
- **Angle-Angle-Side Congruence Postulate (AAS):** Two triangles are congruent if two angles and a non-included side of one triangle are congruent to the corresponding two angles and the corresponding non-included side of the other triangle.



Benchmark Codes: MA.912.G.4.4, MA.912.G.4.6

Example

Is $\triangle ABC$ congruent to $\triangle DEF$?



Compare the sides and angles of the two triangles.

$\overline{AB} \cong \overline{DE}$

$\overline{BC} \cong \overline{EF}$

$\angle B \cong \angle E$

$\angle B$ is the included angle formed by $\overline{AB}$ and $\overline{BC}$.

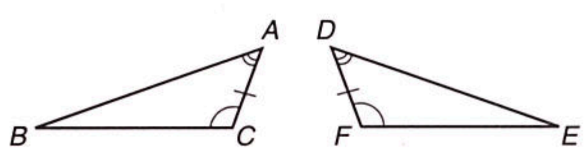
$\angle E$ is the included angle formed by $\overline{DE}$ and $\overline{EF}$.

The **SAS congruence postulate** proves that $\triangle ABC$ is congruent to $\triangle DEF$.

Practice

Directions: For questions 1 through 5, write the postulate that proves the two triangles are congruent. If there is not enough information, write "cannot tell." Show your work.

1.



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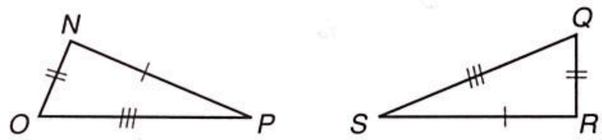
2.



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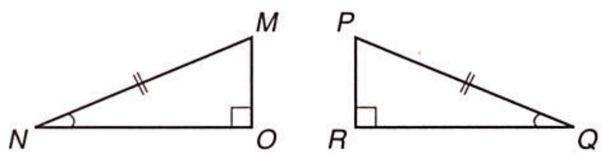
3.



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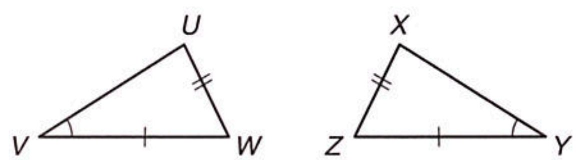

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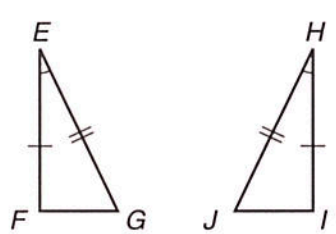

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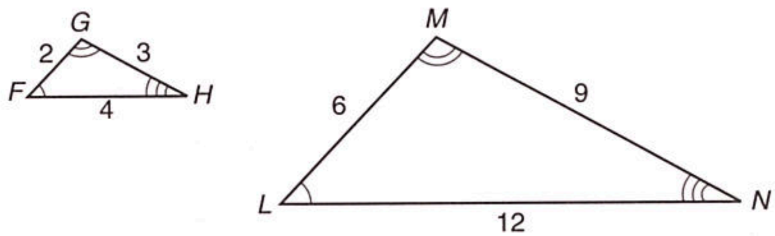
Benchmark Codes: MA.912.G.4.4, MA.912.G.4.6

Similar Triangles

Similar triangles have the same shape but not necessarily the same size. The corresponding angles of similar triangles are congruent, and the lengths of their corresponding sides are proportional. The symbol for similar is $\sim$.



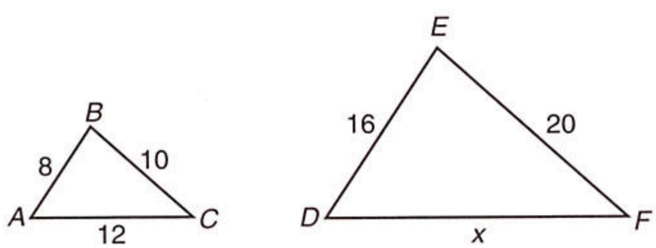
Corresponding angles: $\angle F \cong \angle L$ $\quad \angle G \cong \angle M$ $\quad \angle H \cong \angle N$

Corresponding sides: $\frac{FG}{LM} = \frac{2}{6} = \frac{1}{3}$ $\quad \frac{GH}{MN} = \frac{3}{9} = \frac{1}{3}$ $\quad \frac{FH}{LN} = \frac{4}{12} = \frac{1}{3}$

Therefore, $\triangle FGH \sim \triangle LMN$.

Example

Given: $\triangle ABC \sim \triangle DEF$. What is the value of x?



Set up a proportion, multiply to get cross products, and solve the equation.

$$\frac{8}{16} = \frac{12}{x}$$

$$8 \bullet x = 16 \bullet 12$$

$$8x = 192$$

$$\frac{8x}{8} = \frac{192}{8}$$

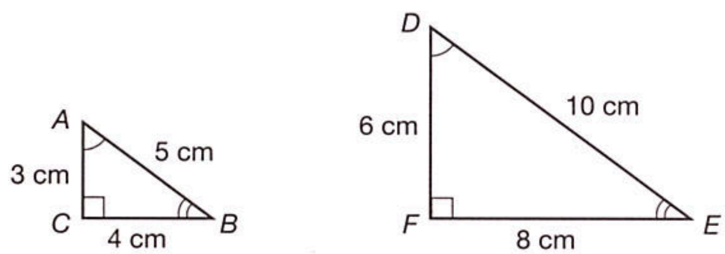
$$x = 24$$

The value of x is 24.

The area of a triangle is equal to the product of the area of a similar triangle and the squared scale factor. The scale factor is the ratio of the triangles' corresponding sides.

Example

Given: $\triangle DEF \sim \triangle ABC$



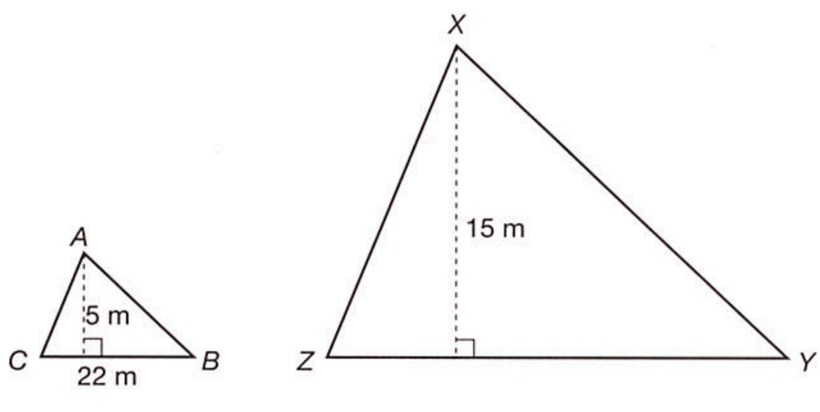
Corresponding sides: $\frac{DE}{AB} = \frac{10}{5} = 2$ $\frac{EF}{BC} = \frac{8}{4} = 2$ $\frac{FD}{CA} = \frac{6}{3} = 2$

Therefore, $\triangle DEF$ has a scale factor of 2 compared to $\triangle ABC$. The area of a triangle is $\frac{1}{2}bh$. The area of $\triangle ABC$ is $\frac{1}{2} \times 3 \text{ cm} \times 4 \text{ cm} = 6 \text{ cm}^2$.

The area of $\triangle DEF$ is equal to the product of the area of the similar triangle and the squared scale factor. The area of $\triangle DEF$ is $6 \text{ cm}^2 \times 2^2 = 24 \text{ cm}^2$.

Example

Given $\triangle ABC \sim \triangle XYZ$, what is the area of $\triangle XYZ$?



The heights of the similar triangles have a ratio of $\frac{15}{5}$. Therefore, $\triangle XYZ$ has a scale factor of 3 compared to $\triangle ABC$.

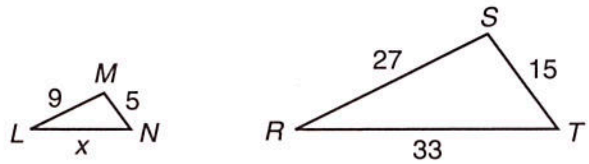
The area of $\triangle ABC$ is $\frac{1}{2} \times 5 \text{ m} \times 22 \text{ m} = 55 \text{ m}^2$. The area of $\triangle XYZ$ is $55 \text{ m}^2 \times 3^2 = 495 \text{ m}^2$.

Benchmark Codes: MA.912.G.4.4, MA.912.G.4.6

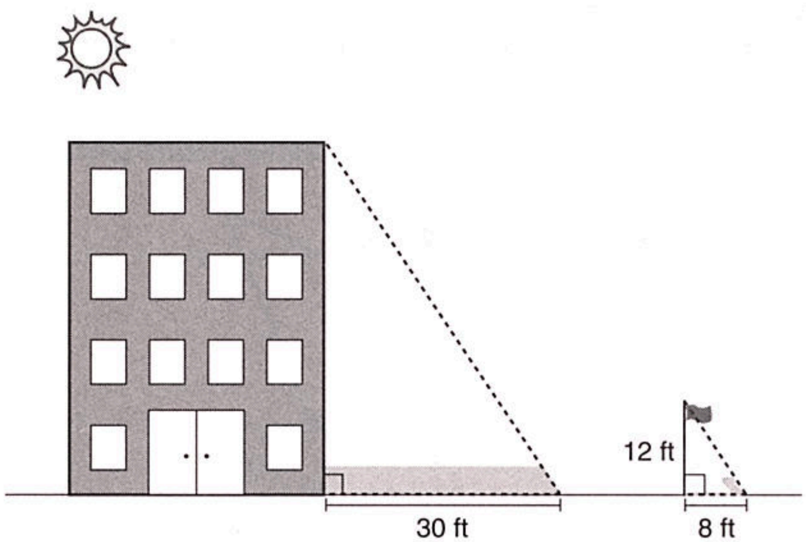
Practice

Directions: Use similar triangles to answer questions 1 through 6.

1. If $\triangle LMN$ is similar to $\triangle RST$, what is the value of x? ____________



2. The following diagram shows the length of the shadows of Crestview High School and a flagpole. The distance from the base of the school to the end of its shadow is 30 feet, and the distance from the flagpole to the end of its shadow is 8 feet. The height of the flagpole is 12 feet. What is the height of Crestview High School?



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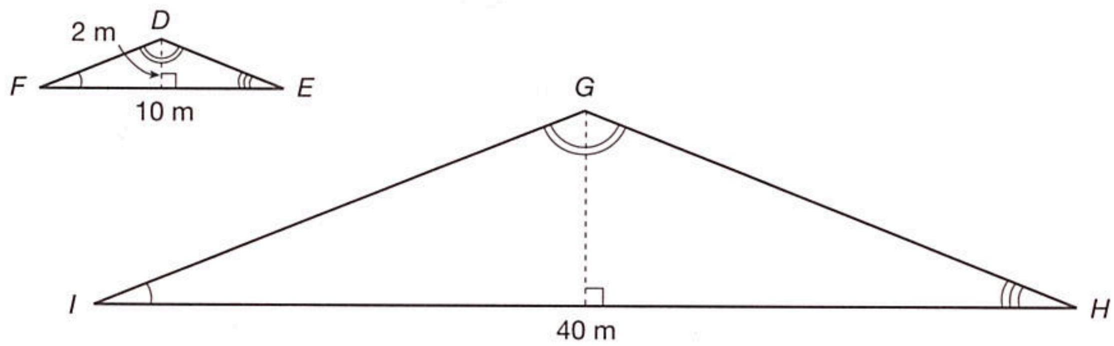
3. An abandoned airfield is in the shape of an equilateral triangle with a side-length of 4,000 feet. A company plans to buy the land and build another triangular airfield that is similar to the old one. The proportion of the side-length of the new airfield to the side-length of the old airfield will be $\frac{4}{5}$. What will be the side-length of the new airfield?

____

4. If $\triangle DEF$ is similar to $\triangle GHI$, what is the area of $\triangle GHI$?



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5. A math teacher creates a right triangle on her computer. The triangle has a base of 12 inches and a height of 5 inches. The teacher then uses a projector to project the image from her computer onto a screen. The triangle on the screen is 8 times larger than the triangle on her computer. What is the area of the triangle on the screen?

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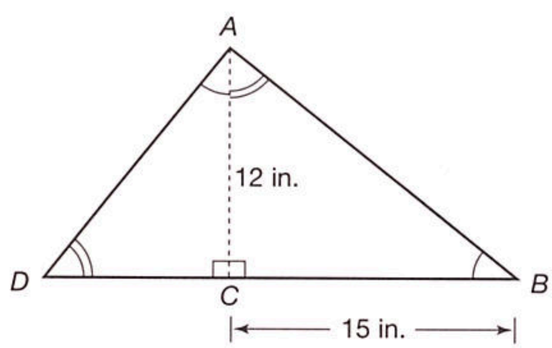
6. If $\triangle ACB$ is similar to $\triangle DCA$, what is the area of $\triangle DCA$?



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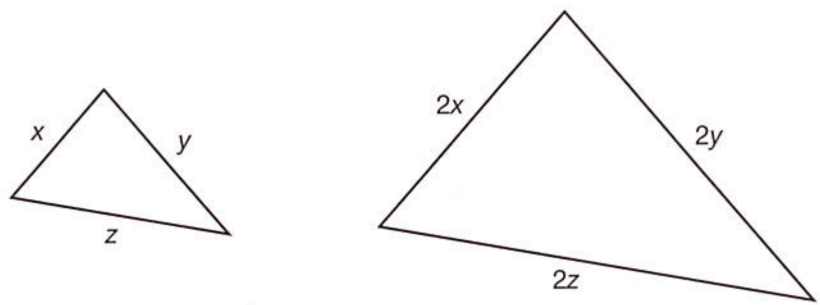
Benchmark Codes: MA.912.G.4.6

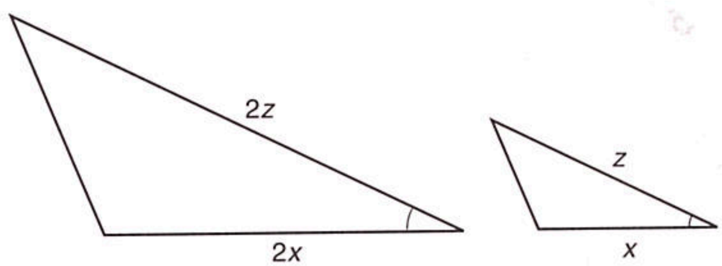
Proving Triangles Similar Using Postulates

It is not necessary to compare all six pairs of corresponding parts of two triangles to prove they are similar. A postulate can be used to prove two triangles are similar by comparing two or three pairs of corresponding parts.

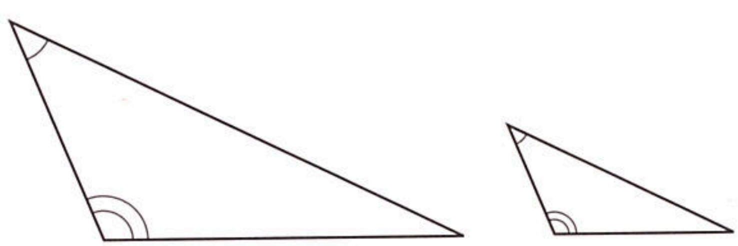
- **Side-Side-Side Similarity Postulate (SSS):** Two triangles are similar if the three sides of one triangle are proportional to the corresponding three sides of the other triangle.



- **Side-Angle-Side Similarity Postulate (SAS):** Two triangles are similar if an angle of one triangle is congruent to an angle of the other triangle, and the corresponding sides that include these angles are proportional.

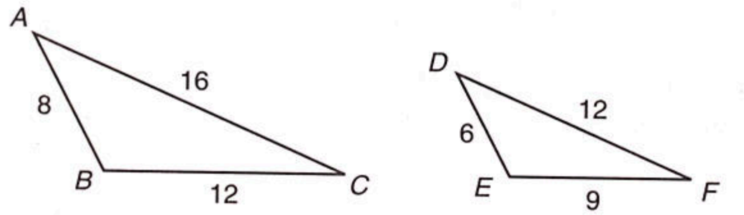


- **Angle-Angle Similarity Postulate (AA):** Two triangles are similar if two angles of one triangle are congruent to the corresponding two angles of the other triangle.



Example

Is $\triangle ABC$ similar to $\triangle DEF$?



Compare the sides and angles of the two triangles.

The drawing does not show whether any of the angles are congruent. The drawing does show that corresponding sides are proportional.

$$\frac{AB}{DE} = \frac{8}{6} = \frac{4}{3}$$

$$\frac{BC}{EF} = \frac{12}{9} = \frac{4}{3}$$

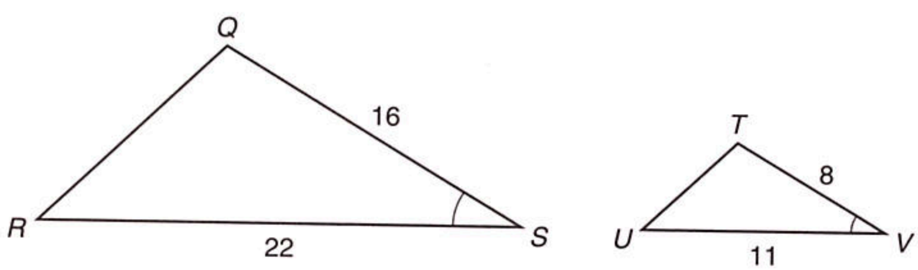
$$\frac{AC}{DF} = \frac{16}{12} = \frac{4}{3}$$

The **SSS similarity postulate** proves that $\triangle ABC$ is similar to $\triangle DEF$.

Practice

Directions: For questions 1 through 5, write the postulate that proves the two triangles are similar. If there is not enough information, write "cannot tell." Show your work.

1.



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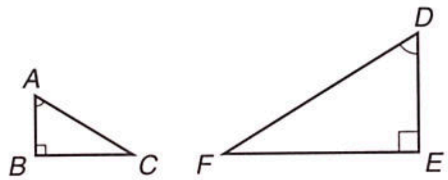
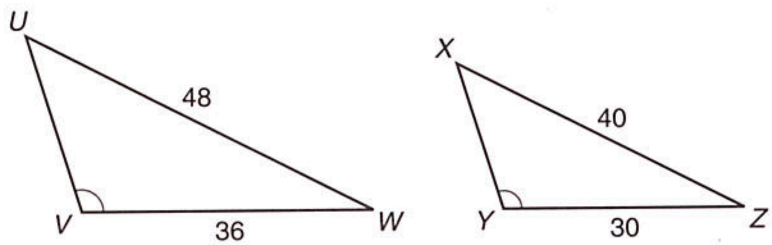
Benchmark Codes: MA.912.G.4.6

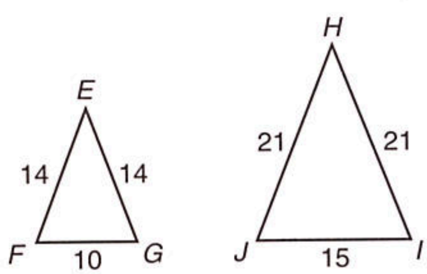
2.



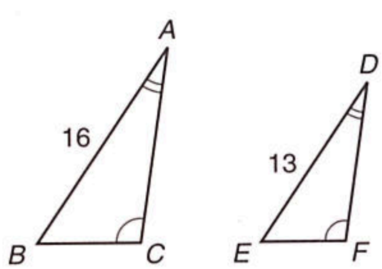
3.



4.

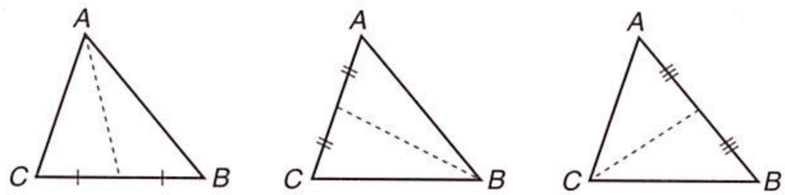


5.

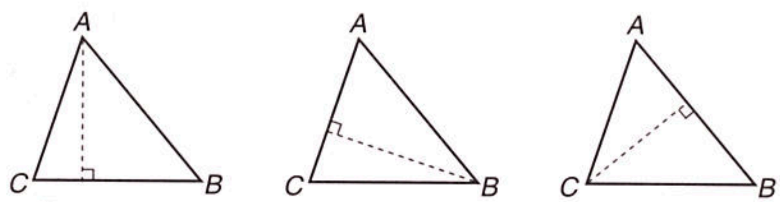


Segments Related to Triangles

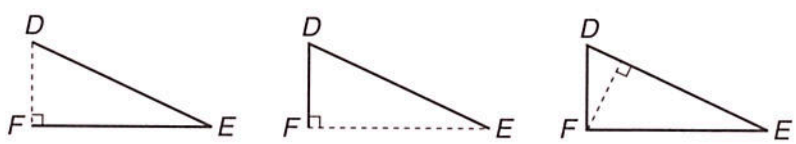
A **median** of a triangle is a segment from a vertex to the midpoint of the opposite side. The dashed lines in the following triangles represent the medians.



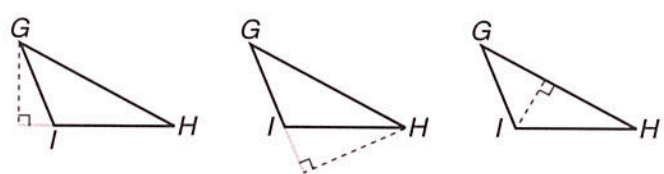
An **altitude** of a triangle is the perpendicular segment from a vertex to its opposite side. In an acute triangle, the three altitudes are all inside the triangle. The dashed lines in the following triangles represent the altitudes.



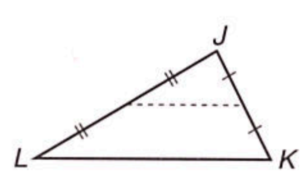
In a right triangle, two of the altitudes are parts of the triangle. They are the legs of the right triangle. The third altitude is inside the triangle.



In an obtuse triangle, two of the altitudes are outside the triangle.



A **midsegment** of a triangle is the segment connecting the midpoints of two sides of a triangle. The midsegment is always parallel to the third side, and the length is half the length of the third side.

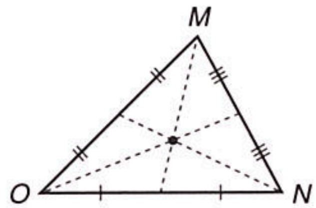


Benchmark Codes: MA.912.G.4.2

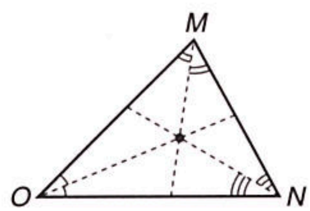
Points of Concurrency Related to Triangles

The term "concurrent" simply means "meeting or intersecting at a point." Therefore, "points of concurrency" refers to the points where segments of a triangle meet.

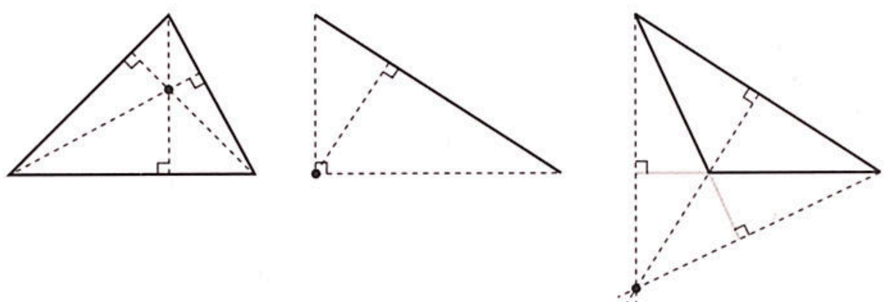
The **centroid** of a triangle is the point where the medians meet.



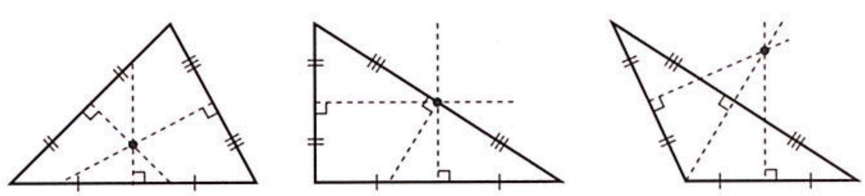
The **incenter** of a triangle is the point where the angle bisectors meet.



The **orthocenter** of a triangle is the point where the altitudes meet.



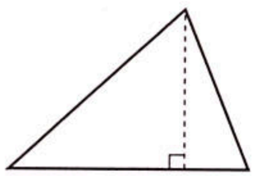
The **circumcenter** of a triangle is the point where the perpendicular bisectors meet.



TIP: The orthocenter and the circumcenter can be located inside, on the border of, or outside the triangle.

Practice

Directions: For questions 1 through 3, write whether the dashed line represents a *median*, an *altitude*, or a *midsegment* of the triangle.

1. 

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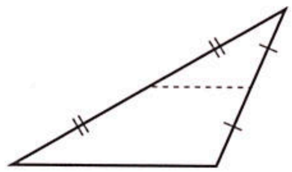
2. 

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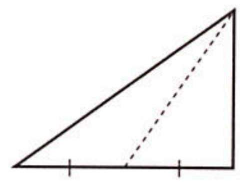
3. 

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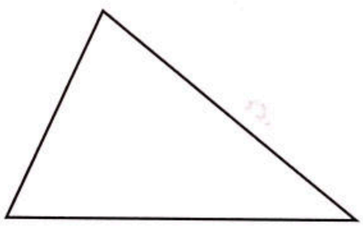
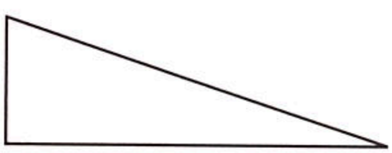
Directions: For questions 4 through 7, find the given point of concurrency of the triangle.

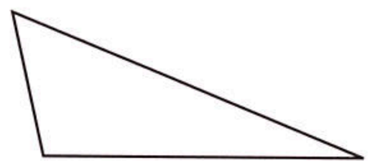
4. centroid



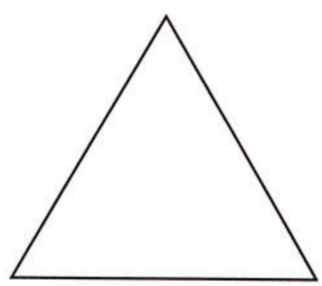
5. orthocenter



6. circumcenter



7. incenter



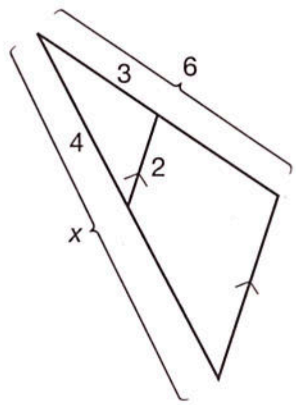
Benchmark Codes: MA.912.G.4.5

Triangle Proportionality Theorem

The triangle proportionality theorem states that a line drawn parallel to any of the sides of a triangle divides the other two sides proportionally. This segment creates two similar triangles. Since a midsegment of a triangle is always parallel to one of the sides, you can apply this theorem to midsegments.

Example

Find the length of x.



Because of the triangle proportionality theorem, the parallel segment forms two similar triangles. Set the ratios of the corresponding sides equal to each other to find the value of x.

$$\frac{3}{6} = \frac{4}{x}$$

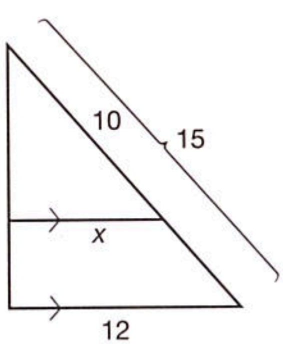
$$3x = 24$$

$$x = 8$$

Practice

Directions: Use the triangle proportionality theorem to answer questions 1 and 2.

1. Find the length of x.



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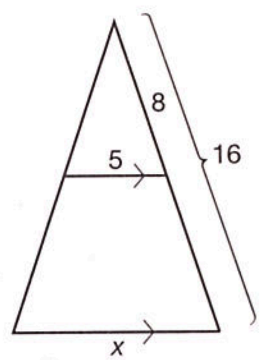
2. Find the length of x.



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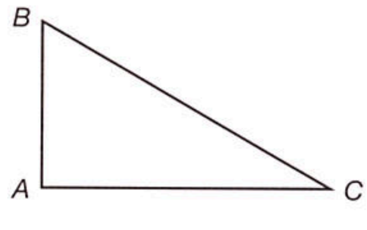
Constructing Congruent Triangles

When given a triangle, you can use a straightedge and a compass to draw a congruent triangle.

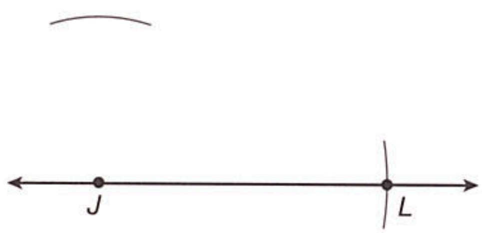
Example

Draw $\triangle JKL$ that is congruent to $\triangle ABC$.

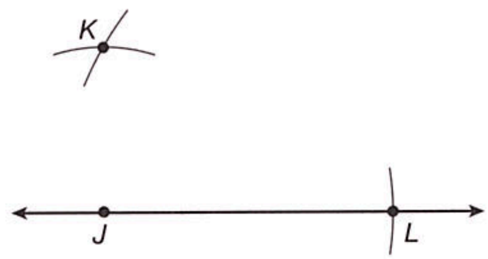


First, draw a point that will be a vertex for $\triangle JKL$. You can label that point J. Draw a line through point J. Then set the width of your compass to the length of AC. Keep the compass width the same, and then draw an arc from point J, that intersects the line. This will be used to form point L.

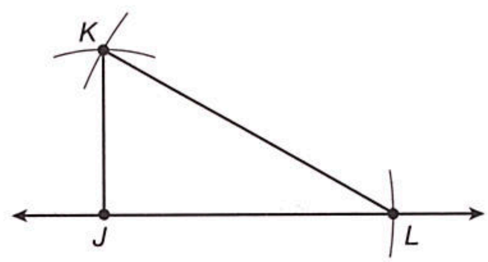
Set the width of your compass to the length of AB. Keep the compass width the same, and then draw an arc from point J.



Set the width of your compass to the length of BC. Keep the compass width the same, and then draw an arc from point L so that it intersects the other arc. This will be used to form point K.



The intersection of the arcs can be labeled point K. Finally, use a straight edge to connect points J, K, and L.

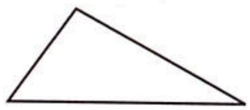


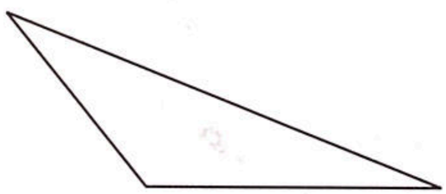
$\triangle JKL \cong \triangle ABC$

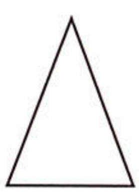
TIP: You can also use a ruler and a protractor to create a congruent triangle. Measure the lengths of the sides and the measures of the angles of the original triangle. Then recreate the triangle using your ruler to create congruent side lengths and your protractor to create congruent interior angles.

Practice

Directions: For questions 1 through 3, use a compass and a straightedge to construct a triangle congruent to the given triangle.

1. 

2. 

3. 

Geometry Practice begins on the following page.