Benchmark Codes: MA.912.G.1.1

Lesson 7: The Coordinate System

In this lesson, you will learn to find the distance between two points and the midpoint of a line segment. You will identify the slope of a line and create parallel or perpendicular lines. You will identify congruent figures and apply transformations. Finally, you will find the equation of a circle in a coordinate system and graph a circle given its equation or its center and radius.

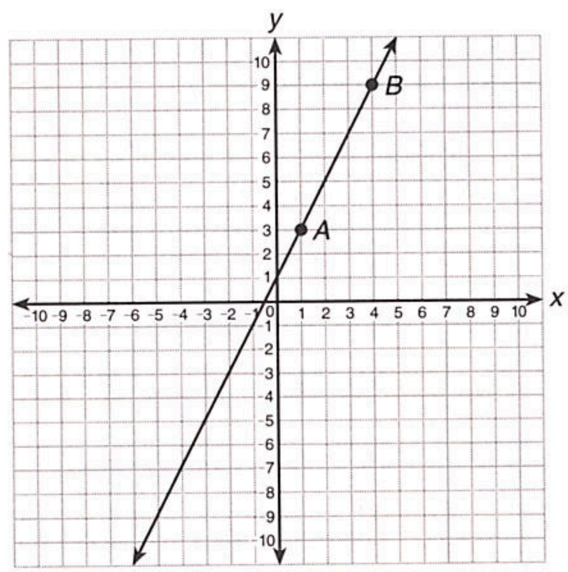
The Distance between Points

The distance between two points is the same as the length of a segment that has the points as its endpoints. If two points have the coordinates (x_1, y_1) and (x_2, y_2), the distance between the points, or the length of the segment between the points, can be found using the following **distance formula**.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example

What is the length of $\overline{AB}$?



Let $A = (x_1, y_1) = (1, 3)$ and $B = (x_2, y_2) = (4, 9)$. Substitute these values into the distance formula and simplify.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(4 - 1)^2 + (9 - 3)^2}$$

$$= \sqrt{45} = 6.7082...$$

The length of $\overline{AB}$ is $\sqrt{45}$ or $3\sqrt{5}$.

Practice

Directions: For question 1 through 6, find the distance between the given points. If necessary, leave your answer in simplest radical form.

1. (−3, 7) and (0, 11)

   $d =$ ________

2. (−7, 3) and (−1, −2)

   $d =$ ________

3. (5, −1) and (−7, 4)

   $d =$ ________

4. (9, 4) and (12, −7)

   $d =$ ________

5. (8, 3) and (−2, 3)

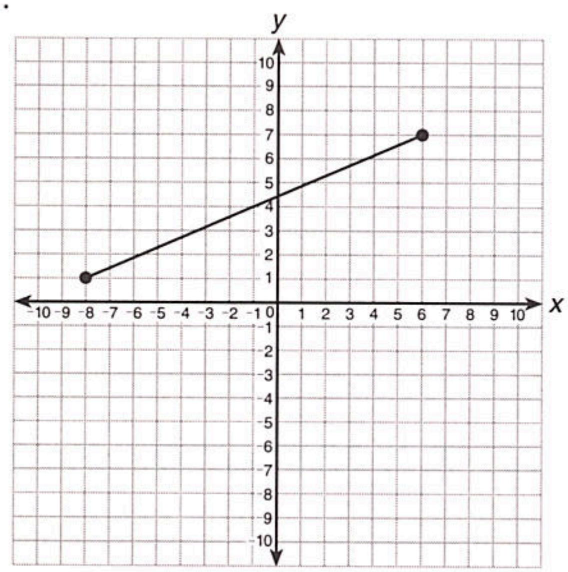
   $d =$ ________

6. (10, −10) and (−10, 10)

   $d =$ ________

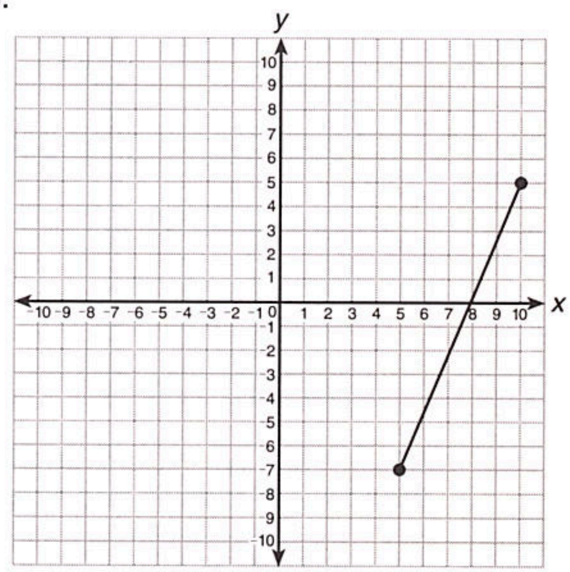
Directions: For questions 7 and 8, find the length of the given line segment. If necessary, leave your answer in simplest radical form.

7.



$d =$ ________

8.



$d =$ ________

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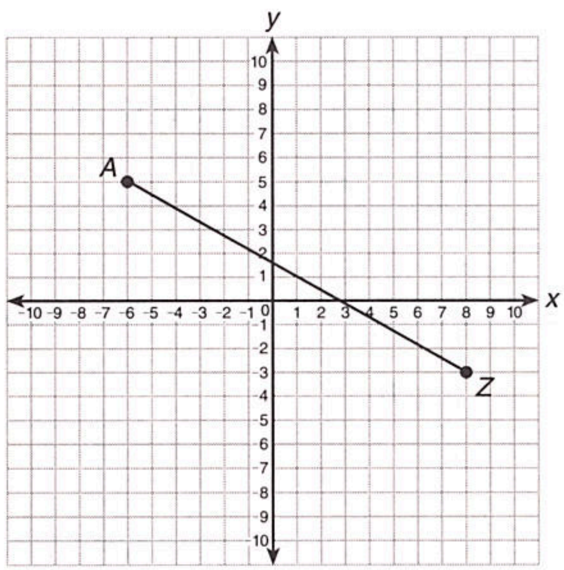
Midpoint

The **midpoint**, ***M***, is the point that divides a segment into two equal segments. Two points have the coordinates (x_1, y_1) and (x_2, y_2). Think of the midpoint as the average of the two endpoints. Use the following formula to find the midpoint of a segment.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example

Find the midpoint of $\overline{AZ}$.



First, find the coordinates of point *A* and point *Z*.

A: (−6, 5) *Z*: (8, −3)

Substitute the values into the midpoint formula and simplify.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{-6 + 8}{2}, \frac{5 + (-3)}{2}\right)$$

$$= \left(\frac{2}{2}, \frac{2}{2}\right)$$

$$= (1, 1)$$

The midpoint of $\overline{AZ}$ is (1, 1).

Practice

Directions: For questions 1 through 6, find the midpoint for the two given points.

1. (−4, 7) and (−7, 4)

   $M =$ ______________

2. (1, −8) and (5, −1)

   $M =$ ______________

3. (0, 3) and (9, 0)

   $M =$ ______________

4. (−4, −8) and (−2, 2)

   $M =$ ______________

5. (5, −9) and (5, −3)

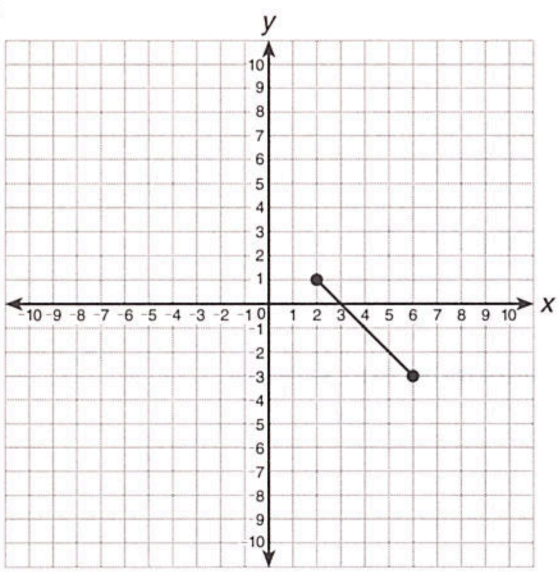
   $M =$ ______________

6. (−4, −10) and (7, 2)

   $M =$ ______________

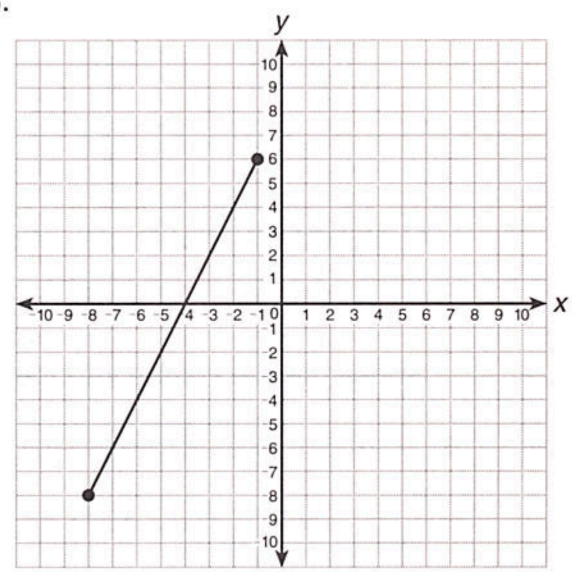
Directions: For questions 7 and 8, find the midpoint of the given line segment.

7.



$M =$ ______________

8.



$M =$ ______________

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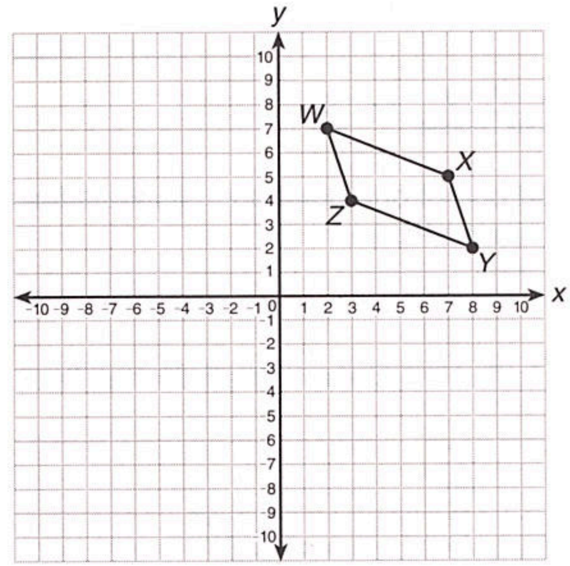
Directions: For questions 9 through 12, use the midpoint formula to find the midpoint of each side of parallelogram *WXYZ*.



9. $\overline{WX}$ ____________________

10. $\overline{WZ}$ ____________________

11. $\overline{XY}$ ____________________

12. $\overline{YZ}$ ____________________

13. If you draw diagonal $\overline{XZ}$, what will be its midpoint?

A. (5, 4)

B. $\left(5\frac{1}{2}, 4\frac{1}{2}\right)$

C. $\left(5, 4\frac{1}{2}\right)$

D. (5, 5)

Slopes of Lines

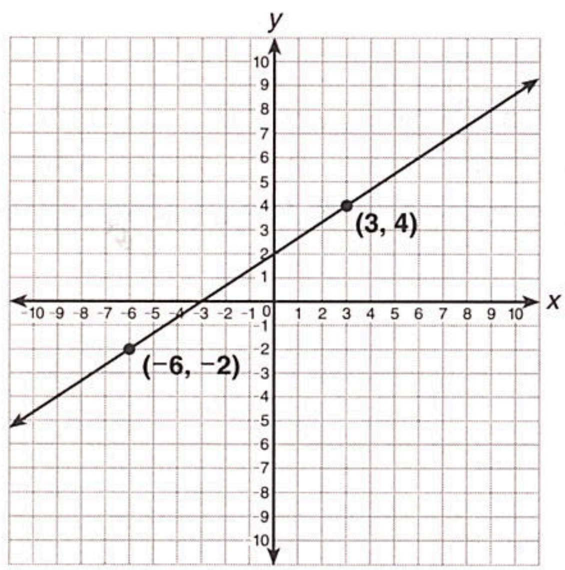
The **slope** of a line describes how steep the change is as you follow the line from left to right. You can find the slope of a line that passes through two given points using the following formula:

$$\textbf{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{vertical change}}{\text{horizontal change}} \text{ or } \frac{\text{rise}}{\text{run}}$$

where (x_1, y_1) is the ordered pair of one point and (x_2, y_2) is the ordered pair of another point. Write the fraction in simplest form.

Example

Use the formula to find the slope of the following line.



(3, 4) and (−6, −2) are the ordered pairs of two points on the line.

$$\underset{x_1\ y_1}{(3,\ 4)} \text{ and } \underset{x_2\ \ y_2}{(-6,\ -2)}$$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 4}{-6 - 3} = \frac{-6}{-9} = \frac{2}{3}$$

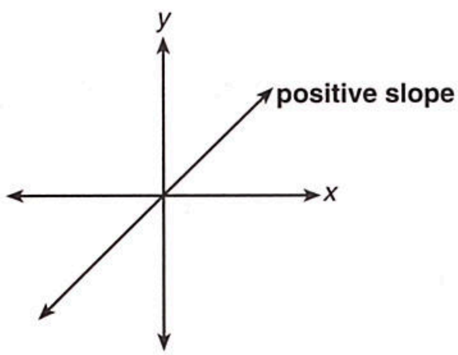
The slope of the line is $\frac{2}{3}$. The line rises as you follow it from left to right.

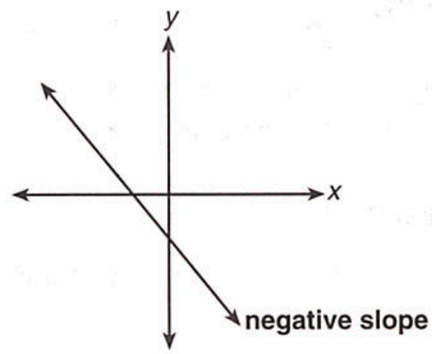
TIP: It does not matter which two points you use to calculate the slope of a line.

Benchmark Codes: MA.912.G.1.1

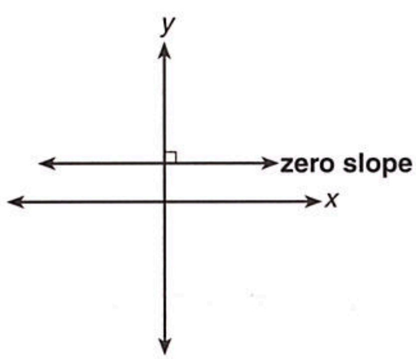
Knowing Slope When You See It

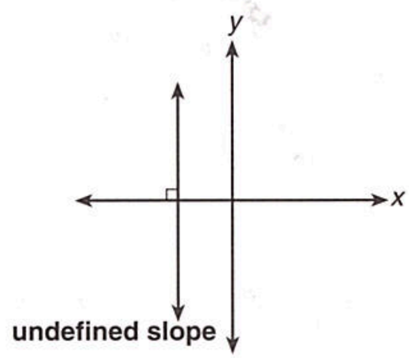
When you look at the graph of a line, you can tell if the line has a positive or negative slope. A line has a positive slope when it rises from left to right and a negative slope when it falls from left to right.





A horizontal line has a slope of zero. This means that there is no vertical change. The slope fraction has a numerator of zero $\left(\text{for example, } \frac{0}{5}\right)$. A vertical line has an undefined slope. This means that there is no horizontal change. The slope fraction has a denominator of zero $\left(\text{for example, } \frac{5}{0}\right)$.





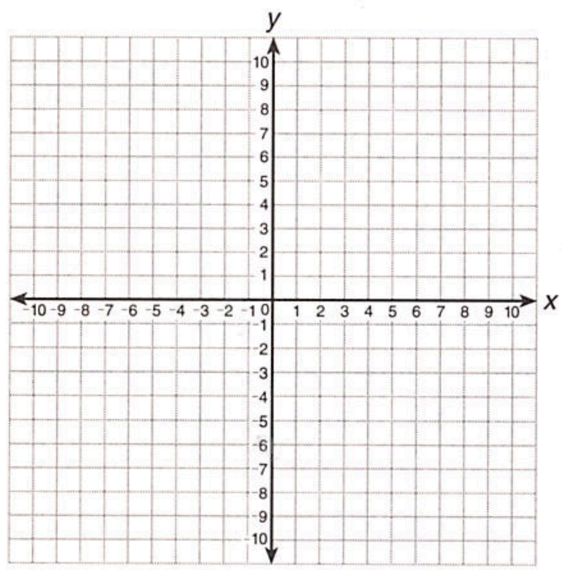
The slope also tells how steep the line is. The larger the absolute value of the slope, the steeper the line. The smaller the absolute value of the slope, the flatter the line.

Practice

Directions: For questions 1 through 4, find the slope of the line that passes through the given points. Then, graph the line and verify that your slope is correct.

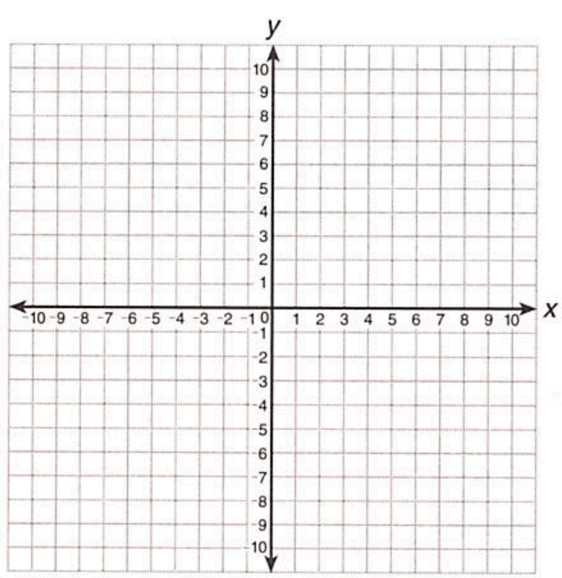
1. (−4, 6) and (3, 4)

   slope __________



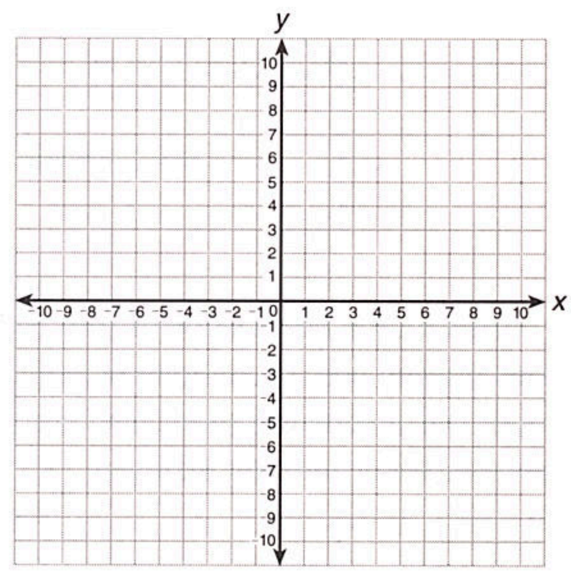
2. (−4, 6) and (−6, −2)

   slope __________



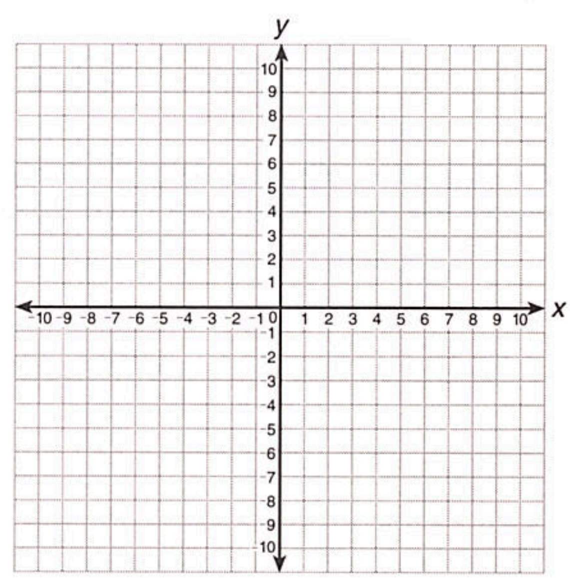
3. (−4, 6) and (4, −3)

   slope __________



4. (3, 4) and (4, −3)

   slope __________



Benchmark Codes: MA.912.G.1.1

Parallel and Perpendicular Lines

The slopes of parallel lines are equal. The slopes of perpendicular lines are opposite reciprocals. The product of opposite reciprocals is -1, so you can multiply the slopes of lines to see if they are perpendicular.

Example

What is the slope of a line that is parallel to $-10x + 5y = 15$?

First, find the slope of $-10x + 5y = 15$ by writing the equation in slope-intercept form ($y = mx + b$, where m is the slope).

$$-10x + 5y = 15$$
$$5y = 10x + 15$$
$$y = 2x + 3$$

The slope of $-10x + 5y = 15$ is 2 ($m = 2$).

Therefore, the slope of a line parallel to $-10x + 5y = 15$ is also 2.

Example

What is the slope of a line that is perpendicular to $77x - 11y = 22$?

First, find the slope of $77x - 11y = 22$.

$$77x - 11y = 22$$
$$-11y = -77x + 22$$
$$y = 7x - 2$$

The slope of $77x - 11y = 22$ is 7. The opposite reciprocal of 7 is $-\frac{1}{7}$.

Therefore, the slope of a line perpendicular to $77x - 11y = 22$ is $-\frac{1}{7}$.

Practice

Directions: For questions 1 through 6, find the slope of a line that is parallel and the slope of a line that is perpendicular to the line with the given linear equation.

1. $-4x - 3y = 9$

   parallel __________

   perpendicular __________

2. $2x + 4y = 3$

   parallel __________

   perpendicular __________

3. $12x - y = 17$

   parallel __________

   perpendicular __________

4. $x = y$

   parallel __________

   perpendicular __________

5. $8x = 3y + 2$

   parallel __________

   perpendicular __________

6. $-2x + y = 5$

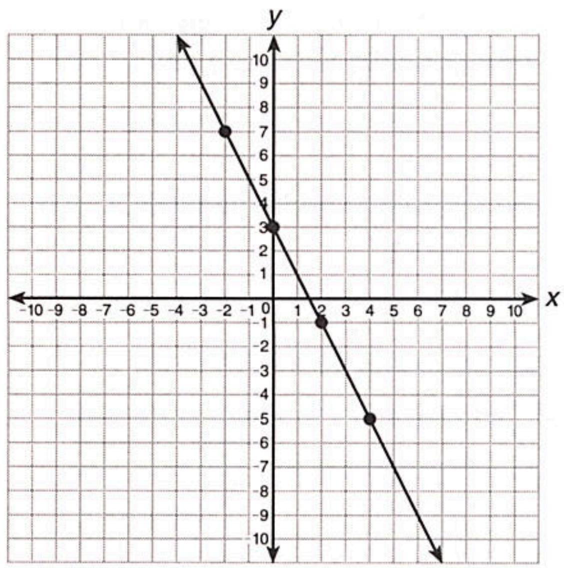
   parallel __________

   perpendicular __________

Benchmark Codes: MA.912.G.1.1

Directions: For questions 7 and 8, draw a line that is parallel to, and a line that is perpendicular to, the given line by plotting specific points. Then, give the slope of each line that you drew.

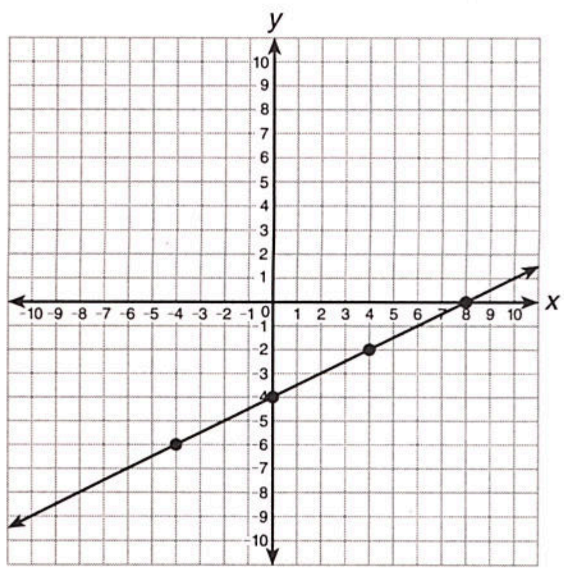
7.



parallel __________

perpendicular __________

8.



parallel __________

perpendicular __________

9. What is the slope of a line that is parallel to $-18x + 2y = 6$?

A. -9

B. 6

C. 9

D. 24

10. What is the slope of a line that is perpendicular to $10x - 20y = 0$?

A. -2

B. 0

C. $\frac{1}{2}$

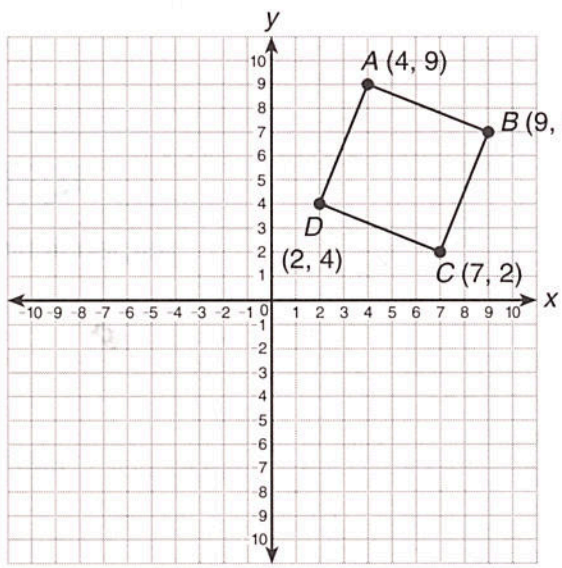
D. 2

Identifying Geometric Figures

You can use the distance formula, midpoint formula, and slope formula to identify geometric figures on a coordinate plane. For example, to verify that a quadrilateral is a parallelogram, you can use the slope formula to determine that opposite sides are parallel and the distance formula to determine that opposite sides are congruent.

Example

Use the distance formula and the slope formula to verify that *ABCD* is a square.



Use the distance formula to find the length of each side.

$$AB = \sqrt{(9-4)^2 + (7-9)^2} = \sqrt{29}$$

$$BC = \sqrt{(7-9)^2 + (2-7)^2} = \sqrt{29}$$

$$CD = \sqrt{(2-7)^2 + (4-2)^2} = \sqrt{29}$$

$$AD = \sqrt{(2-4)^2 + (4-9)^2} = \sqrt{29}$$

The side lengths are all congruent. You need to determine whether the opposite sides are parallel and whether the angles are right angles. The next page shows you how.

Benchmark Codes: MA.912.G.3.3

Use the slope formula to find the slope of each segment.

slope of $\overline{AB} = \frac{7-9}{9-4} = -\frac{2}{5}$

slope of $\overline{BC} = \frac{2-7}{7-9} = \frac{5}{2}$

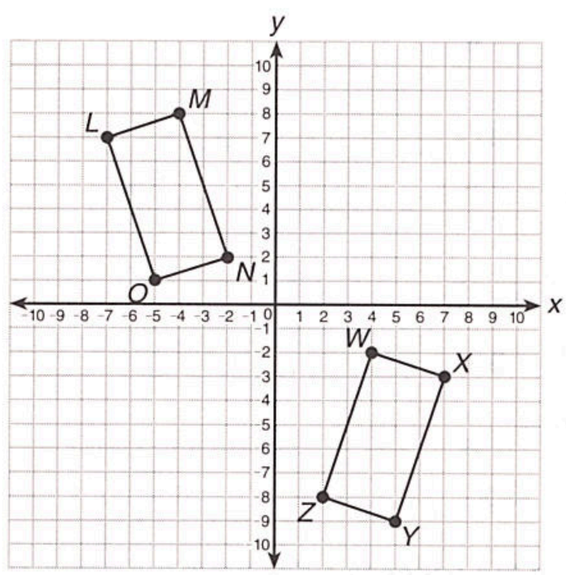
slope of $\overline{CD} = \frac{4-2}{2-7} = -\frac{2}{5}$

slope of $\overline{AD} = \frac{4-9}{2-4} = \frac{5}{2}$

Since $\overline{AB}$ and $\overline{CD}$ have the same slope, they are parallel. Likewise, $\overline{BC}$ and $\overline{AD}$ have the same slope, so they are also parallel. The slopes of $\overline{AB}$ and $\overline{CD}$ are reciprocals of the slopes of $\overline{BC}$ and $\overline{AD}$ and are also opposite in sign, so $\overline{AB}$ and $\overline{CD}$ are each perpendicular to both $\overline{BC}$ and $\overline{AD}$. Therefore, *ABCD* is a square.

Example

Use the distance formula and the slope formula to verify that *LMNO* and *WXYZ* are congruent rectangles.



Two rectangles are congruent if corresponding sides are congruent. Therefore, you must first show that the corresponding sides are congruent and then show that each quadrilateral is a rectangle.

Use the distance formula to find the length of each side of each quadrilateral.

$$LM = \sqrt{(-4 - (-7))^2 + (8 - 7)^2} = \sqrt{10}$$
$$MN = \sqrt{(-2 - (-4))^2 + (2 - 8)^2} = \sqrt{40}$$
$$NO = \sqrt{(-5 - (-2))^2 + (1 - 2)^2} = \sqrt{10}$$
$$OL = \sqrt{(-7 - (-5))^2 + (7 - 1)^2} = \sqrt{40}$$

$$WX = \sqrt{(7 - 4)^2 + (-3 - (-2))^2} = \sqrt{10}$$
$$XY = \sqrt{(5 - 7)^2 + (-9 - (-3))^2} = \sqrt{40}$$
$$YZ = \sqrt{(2 - 5)^2 + (-8 - (-9))^2} = \sqrt{10}$$
$$ZW = \sqrt{(4 - 2)^2 + (-2 - (-8))^2} = \sqrt{40}$$

The opposite sides of each quadrilateral are congruent. Also, note that the corresponding sides of the two quadrilaterals are congruent. You now need to determine whether the opposite sides of each quadrilateral are parallel, and whether all of the angles are right angles. Use the slope formula to find the slope of each segment of each quadrilateral.

slope of $\overline{LM} = \frac{8 - 7}{-4 - (-7)} = \frac{1}{3}$

slope of $\overline{MN} = \frac{2 - 8}{-2 - (-4)} = -3$

slope of $\overline{NO} = \frac{1 - 2}{-5 - (-2)} = \frac{1}{3}$

slope of $\overline{OL} = \frac{7 - 1}{-7 - (-5)} = -3$

slope of $\overline{WX} = \frac{-3 - (-2)}{7 - 4} = -\frac{1}{3}$

slope of $\overline{XY} = \frac{-9 - (-3)}{5 - 7} = 3$

slope of $\overline{YZ} = \frac{-8 - (-9)}{2 - 5} = -\frac{1}{3}$

slope of $\overline{ZW} = \frac{-2 - (-8)}{4 - 2} = 3$

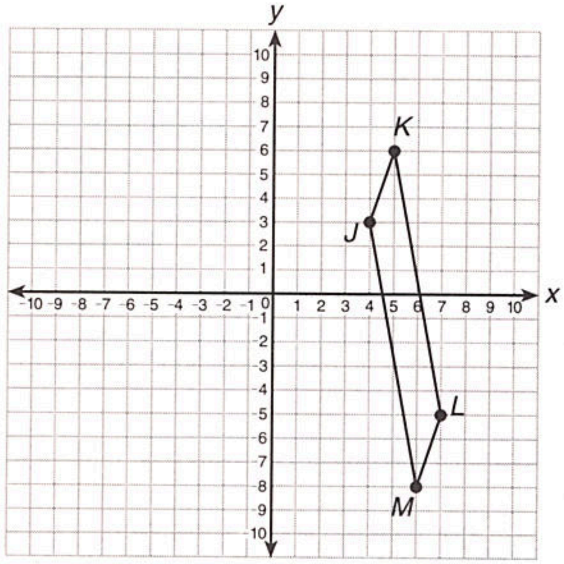
For each quadrilateral, since the opposite sides have the same slope, they are parallel. Thus, each quadrilateral is a parallelogram. Since the adjacent sides of each quadrilateral have slopes that are opposite reciprocals, they are perpendicular. Therefore, each angle in each quadrilateral is right. This means that each quadrilateral is a rectangle. Since the corresponding sides of the two rectangles are congruent, they are congruent rectangles.

Benchmark Codes: MA.912.G.3.3

Practice

Directions: For questions 1 through 3, use the distance formula and the slope formula to identify the given geometric figures. Leave your answers in radical form.

1.



$JK =$ ________

$KL =$ ________

$LM =$ ________

$JM =$ ________

slope of $\overline{JK}$ = ________

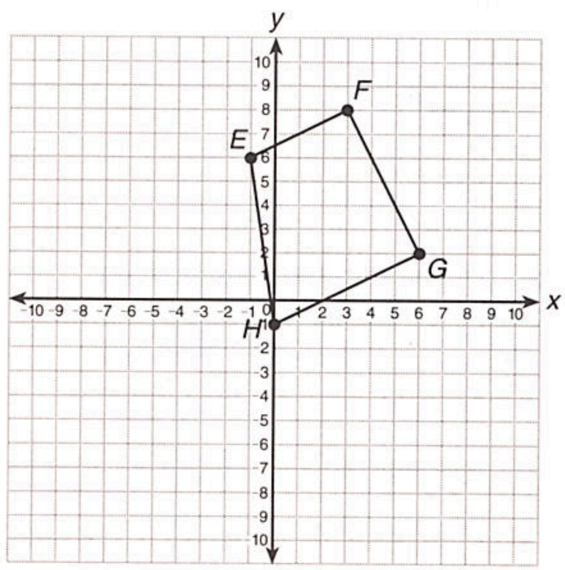
slope of $\overline{KL}$ = ________

slope of $\overline{LM}$ = ________

slope of $\overline{JM}$ = ________

type of quadrilateral: ____________________

2.



$EF =$ ________

$FG =$ ________

$GH =$ ________

$EH =$ ________

slope of $\overline{EF}$ = ________

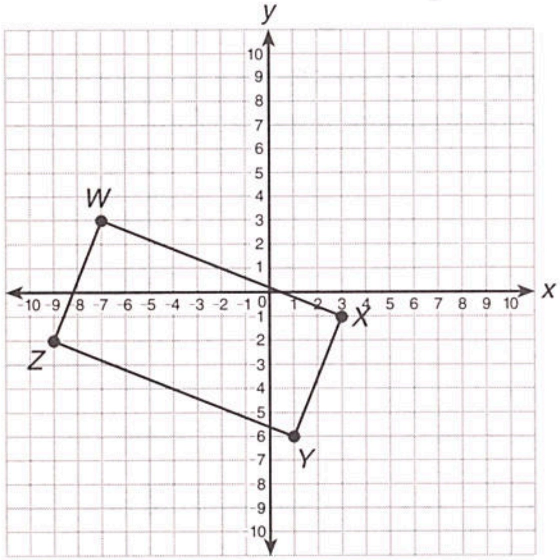
slope of $\overline{FG}$ = ________

slope of $\overline{GH}$ = ________

slope of $\overline{EH}$ = ________

type of quadrilateral: ____________________

3.



$WX =$ __________

$XY =$ __________

$YZ =$ __________

$WZ =$ __________

slope of $\overline{WX}$ = __________

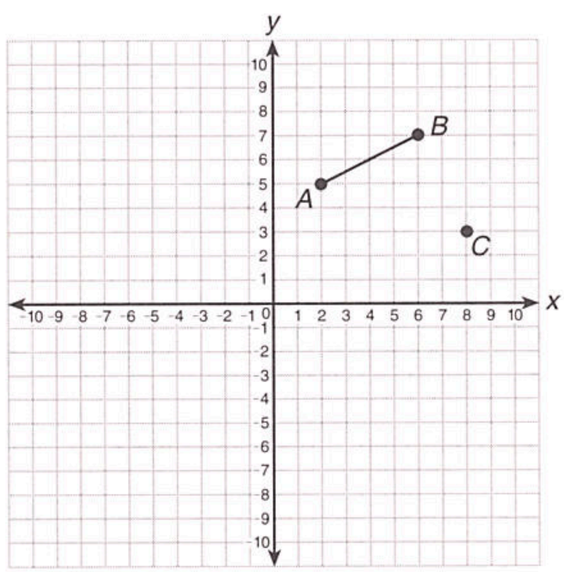
slope of $\overline{XY}$ = __________

slope of $\overline{YZ}$ = __________

slope of $\overline{WZ}$ = __________

type of quadrilateral: ______________________________

4. Kiara is trying to draw a square on the grid below. When she draws the fourth vertex, D, what will be the slope of $\overline{AD}$? Justify your answer.



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Benchmark Codes: MA.912.G.2.4, MA.912.G.3.3

Transformations

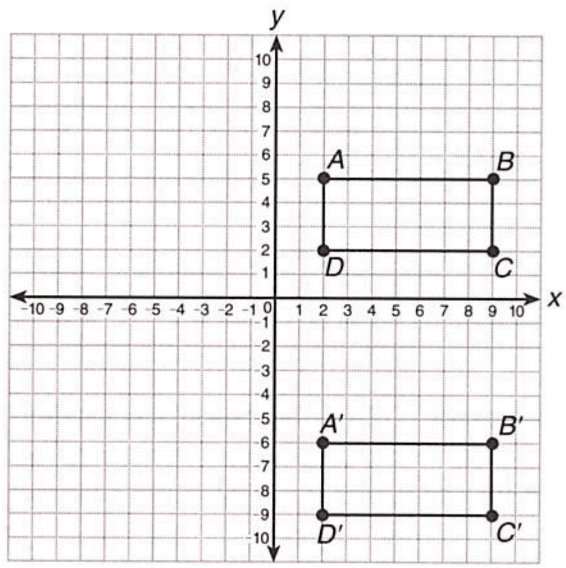
A **transformation** is any operation on a figure by which another image is created. Transformations include reflections (flips), translations (slides), rotations (turns) and dilations.

Translations

A **translation** slides a figure in any direction. A translated figure is congruent to the original figure.

Example

On the following coordinate plane, *ABCD* has been translated 11 units down. The translated figure, *A′B′C′D′*, is congruent to *ABCD*.



The ordered pairs of the vertices of *ABCD* are used as a starting point to translate the rectangle 11 units down. Look carefully at the coordinates of the vertices of *ABCD* and *A′B′C′D′*.

A: (2, 5)	*A′*: (2, −6)
B: (9, 5)	*B′*: (9, −6)
C: (9, 2)	*C′*: (9, −9)
D: (2, 2)	*D′*: (2, −9)

Notice each *y*-coordinate of *A′B′C′D′* is 11 less than its corresponding *y*-coordinate of *ABCD*. The *x*-coordinates remain the same.

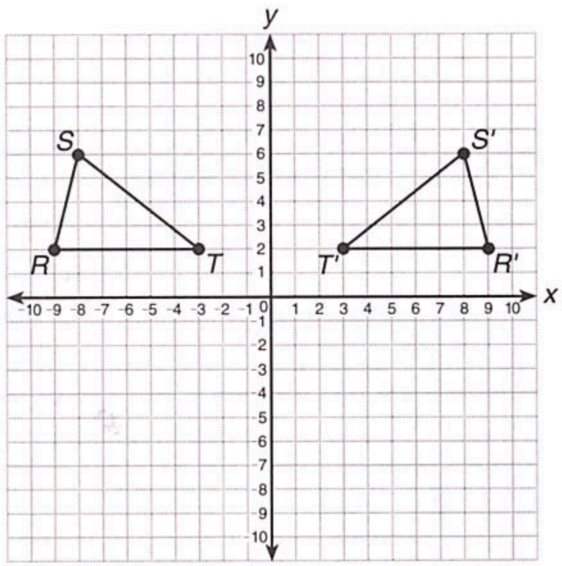
TIP: Translations add a constant number to one or both of the coordinates of every point of a figure.

Reflections

A **reflection** flips a figure over a line. The line is called the **line of reflection**. A reflected figure is congruent to the original figure.

Example

On the following coordinate plane, $\triangle RST$ has been reflected over the *y*-axis. The reflected figure, $\triangle R'S'T'$, is congruent to $\triangle RST$.



The ordered pairs of the vertices of $\triangle RST$ are used as a starting point to reflect the triangle over the *y*-axis. Look carefully at the coordinates of the vertices of $\triangle RST$ and $\triangle R'S'T'$.

R: $(-9, 2)$	R': $(9, 2)$
S: $(-8, 6)$	S': $(8, 6)$
T: $(-3, 2)$	T': $(3, 2)$

Notice each *x*-coordinate of $\triangle R'S'T'$ is the opposite of each *x*-coordinate of $\triangle RST$. This is because the *y*-axis was used as the line of reflection. The *y*-coordinates remain the same.

TIP: If the *x*-axis is used as the line of reflection, then the *y*-coordinates of the figures will be opposites and the *x*-coordinates will be the same.

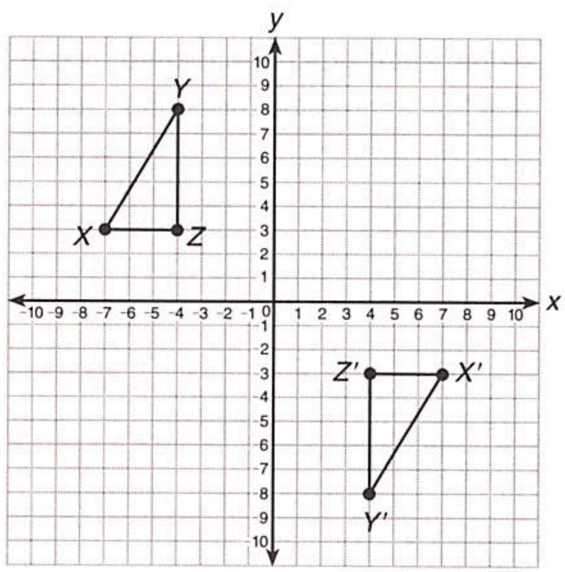
Benchmark Codes: MA.912.G.2.4, MA.912.G.3.3

Rotations

A **rotation** turns a figure around a point. The point is called the **center of rotation**. You will use the origin as the center of rotation in most situations. Figures are usually rotated 90°, 180°, 270°, or 360°. A figure can be rotated in either a clockwise or a counterclockwise direction. A rotated figure is congruent to the original figure.

Example

On the following coordinate plane, $\triangle XYZ$ has been rotated 180° around the origin. The rotated figure, $\triangle X'Y'Z'$, is congruent to $\triangle XYZ$.



The ordered pairs of the vertices of $\triangle XYZ$ are used as a starting point to rotate the triangle around the origin. Look carefully at the coordinates of the vertices of $\triangle XYZ$ and $\triangle X'Y'Z'$.

X: (−7, 3) X': (7, −3)

Y: (−4, 8) Y': (4, −8)

Z: (−4, 3) Z': (4, −3)

Notice each x- and y-coordinate of $\triangle X'Y'Z'$ is the opposite of its corresponding x- and y-coordinate of $\triangle XYZ$.

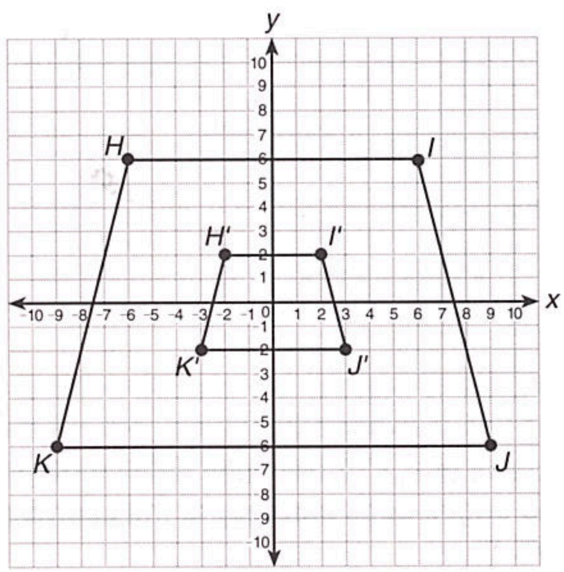
TIP: Some figures are rotated around a point in the figure. The coordinates of that point will not change after the rotation.

Dilations

A **dilation** either enlarges or reduces a figure. The figure is enlarged or reduced from a point called the **center of dilation**. The coordinates of the vertices of the figure are multiplied by a positive number called the **scale factor**. If the scale factor is less than 1, the dilated figure will be a **reduction** of the original figure. If the scale factor is greater than 1, the dilated figure will be an **enlargement** of the original figure. A dilated figure will be similar, but not congruent, to the original figure. The side lengths of a dilated figure are changed by the scale factor. The area of a dilated figure is changed by the scale factor squared.

Example

On the following coordinate plane, *HIJK* has been dilated by a scale factor of $\frac{1}{3}$. The dilated figure, $H'I'J'K'$, is similar, but not congruent, to *HIJK*.



The ordered pairs of the vertices of *HIJK* are used as a starting point to dilate by a scale factor of $\frac{1}{3}$. Look carefully at the coordinates of the vertices of *HIJK* and $H'I'J'K'$.

H: $(-6, 6)$	H': $(-2, 2)$
I: $(6, 6)$	I': $(2, 2)$
J: $(9, -6)$	J': $(3, -2)$
K: $(-9, -6)$	K': $(-3, -2)$

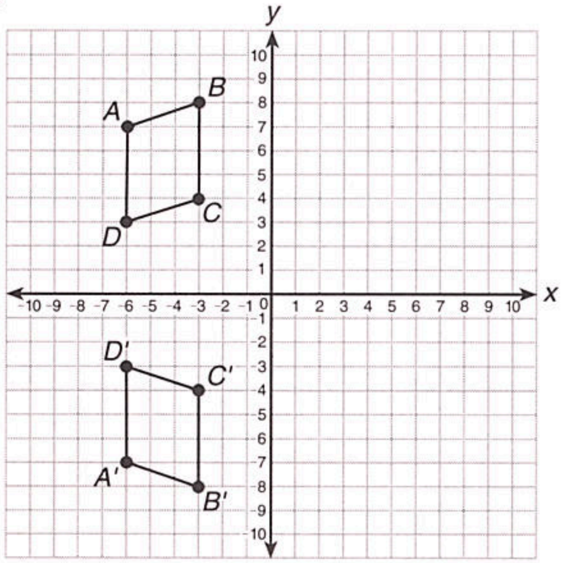
Notice each *x*- and *y*-coordinate of $H'I'J'K'$ is $\frac{1}{3}$ times its corresponding *x*- and *y*-coordinate of *HIJK*.

Benchmark Codes: MA.912.G.2.4, MA.912.G.3.3

Practice

Directions: For questions 1 through 4, determine whether the graph shows a translation, reflection, rotation, or dilation. Determine if the polygons are congruent, similar, or neither.

1.

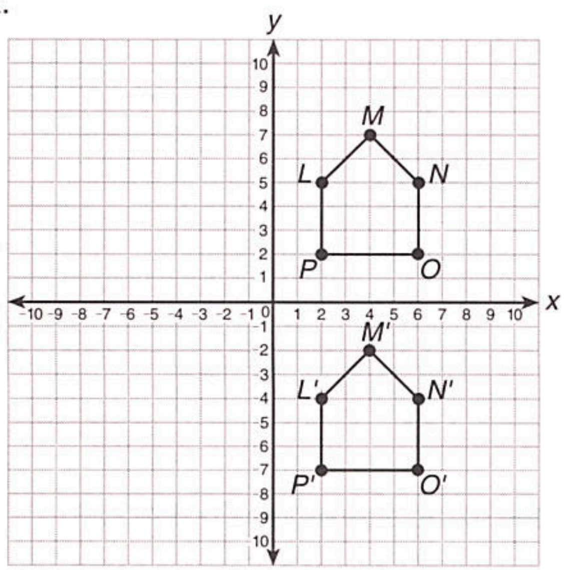


The graph shows a _____________.

Congruent, similar, or neither:

____

2.

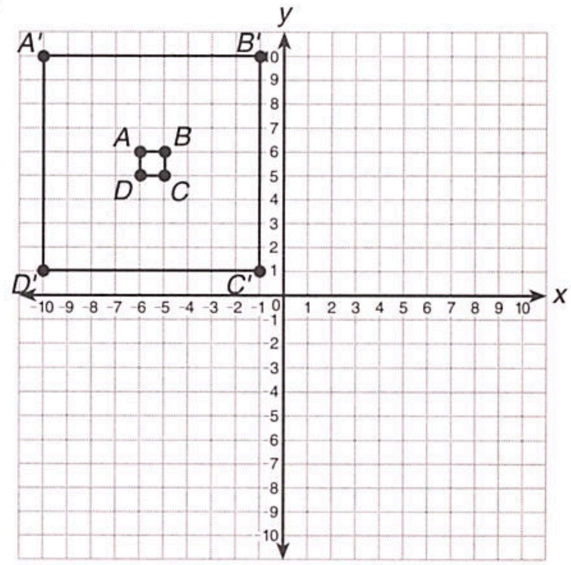


The graph shows a _____________.

Congruent, similar, or neither:

____

3.

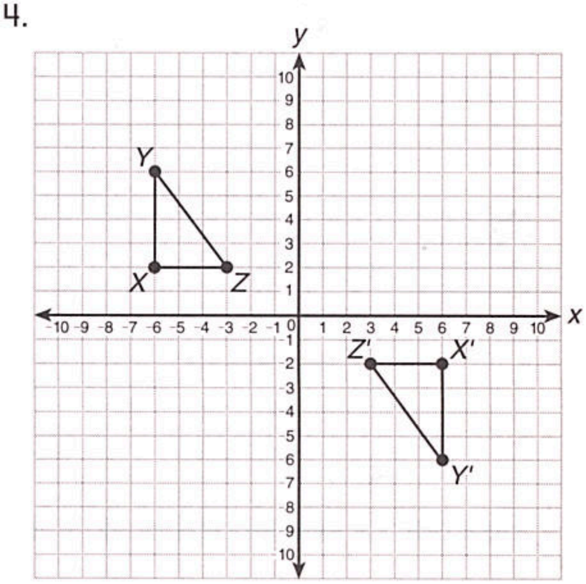


The graph shows a _____________.

Congruent, similar, or neither:

____

4.

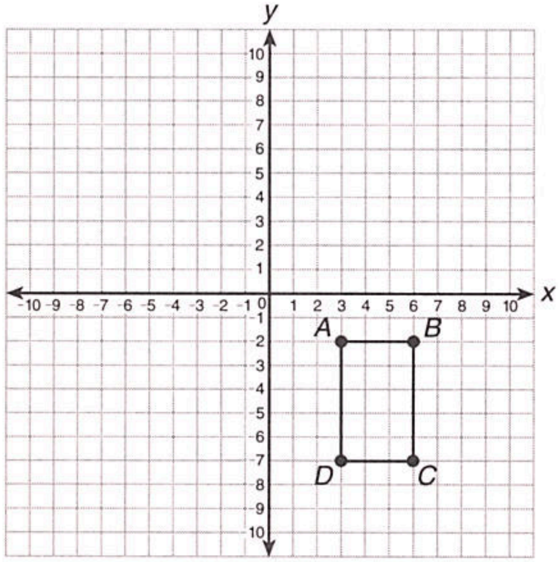


The graph shows a _____________.

Congruent, similar, or neither:

____

5. If *ABCD* is reflected over the *y*-axis, what are the coordinates of the reflected figure *A′B′C′D′*?



A′: ___________

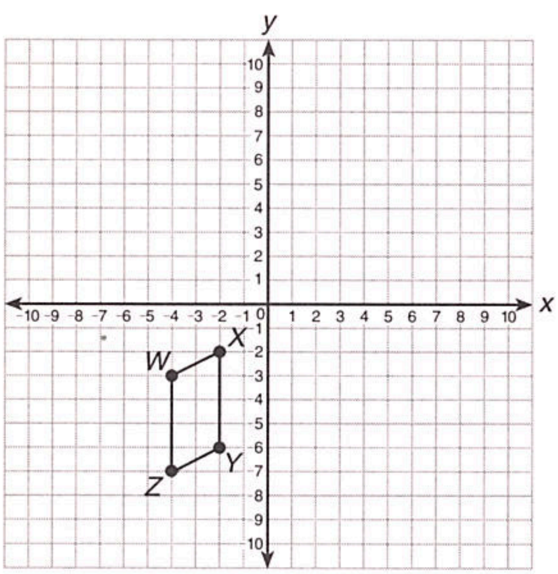
B′: ___________

C′: ___________

D′: ___________

Is *A′B′C′D′* congruent to or similar to *ABCD*? ___________________

6. If *WXYZ* is rotated 90° clockwise around the origin, what are the coordinates of the rotated figure *W′X′Y′Z′*?



W′: ___________

X′: ___________

Y′: ___________

Z′: ___________

How are the side lengths of *W′X′Y′Z′* related to the side lengths of *WXYZ*?

____

How is the area of *W′X′Y′Z′* related to the area of *WXYZ*?

____

Benchmark Codes: MA.912.G.6.6, MA.912.G.6.7

Equations of Circles

A circle is the set of all points in a plane that are an equal distance from a given point. The distance is the radius of the circle and the given point is the center of the circle. The equation of a circle is derived from the distance formula. The following is the equation of a circle with center (h, k) and radius r.

$$(x - h)^2 + (y - k)^2 = r^2$$

To graph a circle, first find (h, k) and r. Then, plot (h, k) and some points that are r units from (h, k). The easiest points to plot are found by adding r to or subtracting r from the x-coordinate of the center and then the y-coordinate of the center:
$(h + r, k)$, $(h - r, k)$, $(h, k + r)$, and $(h, k - r)$.

Example

Find the center and radius of the circle with the following equation.

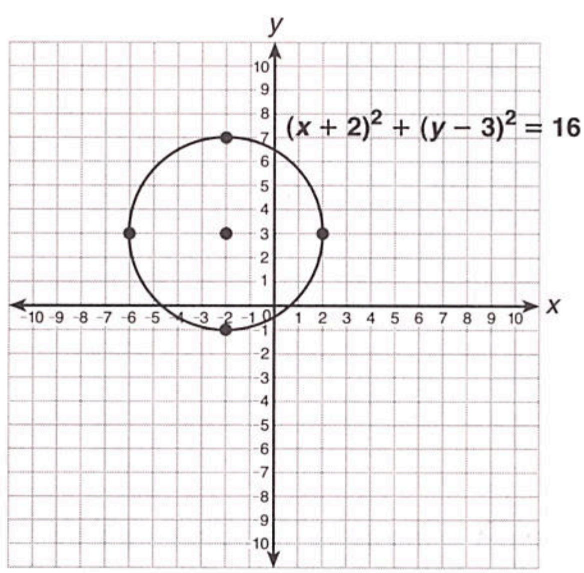
$$(x + 2)^2 + (y - 3)^2 = 16$$

Put the equation in the form shown above.

$$[x - (-2)]^2 + (y - 3)^2 = 4^2$$

Therefore, $(h, k) = (-2, 3)$ and $r = 4$.

To graph the circle, plot a point at $(-2, 3)$. Plot some other points of the circle that are each 4 units from $(-2, 3)$. The following points are each 4 units from $(-2, 3)$: $(2, 3)$, $(-6, 3)$, $(-2, 7)$, and $(-2, -1)$.



TIP: The center of a circle is the midpoint of any diameter.

Example

Find the equation of a circle with a center of (2, 4) and a radius of 3. Graph it on a coordinate grid.

Substitute the given values into the equation of a circle: $(h, k) = (2, 4)$, and $r = 3$.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 2)^2 + (y - 4)^2 = (3)^2$$

$$(x - 2)^2 + (y - 4)^2 = 9$$

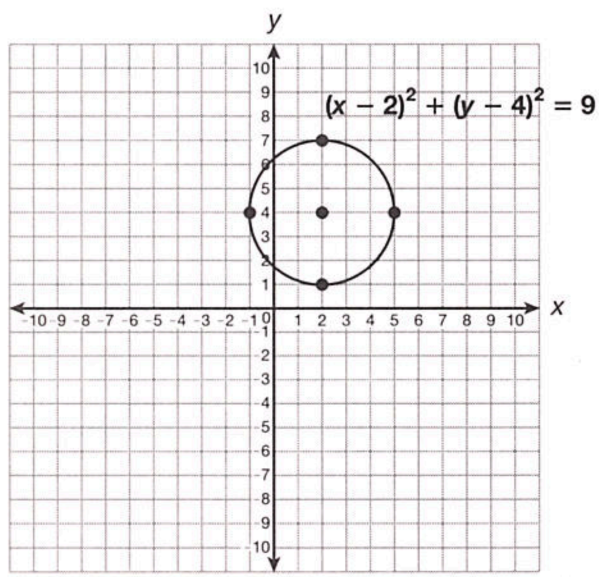
Therefore, the equation of the circle is $(x - 2)^2 + (y - 4)^2 = 9$.

To graph the circle, plot the center point, (2, 4). Then plot other points of the circle that are each 3 units from (2, 4). Adding *r* to and subtracting *r* from the *x*-coordinate and then the *y*-coordinate of the center results in the following points:

$(2 + 3, 4) = (5, 4)$ $(2 - 3, 4) = (-1, 4)$

$(2, 4 + 3) = (2, 7)$ $(2, 4 - 3) = (2, 1)$

The center and four plotted points are shown.



TIP: Because every point on a circle is the same distance from its center, the equation for a circle is related to the distance formula. (See page 135.) The difference is that *r* is used instead of *d* in the equation for a circle and each side of the equation is squared.

Benchmark Codes: MA.912.G.6.6, MA.912.G.6.7

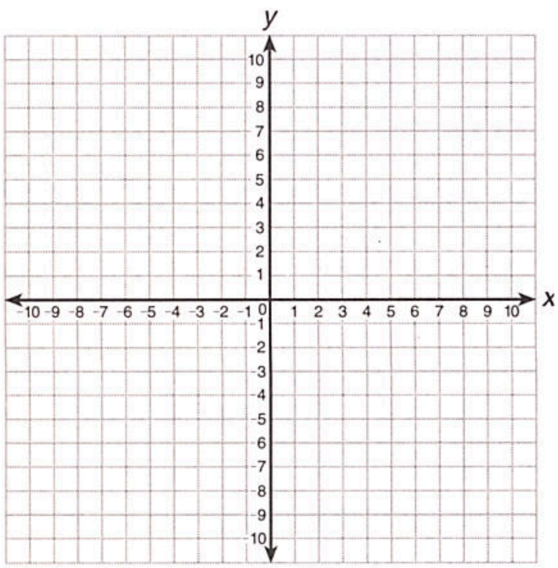
Practice

Directions: For questions 1 through 4, find the center and radius of the circle with the given equation. Then, graph the circle.

1. $(x - 5)^2 + (y + 1)^2 = 1$

   center __________

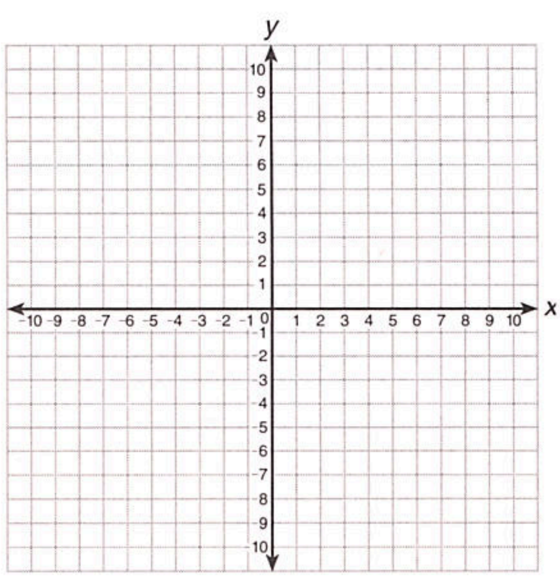
   radius __________



2. $(x - 3)^2 + y^2 = 9$

   center __________

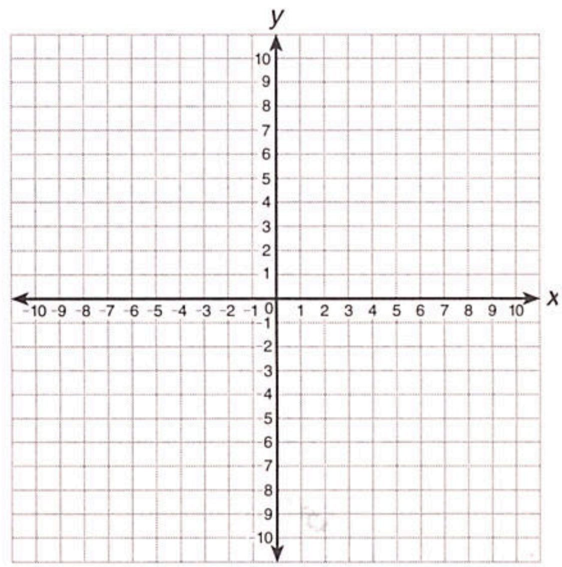
   radius __________



3. $x^2 + y^2 = 25$

   center __________

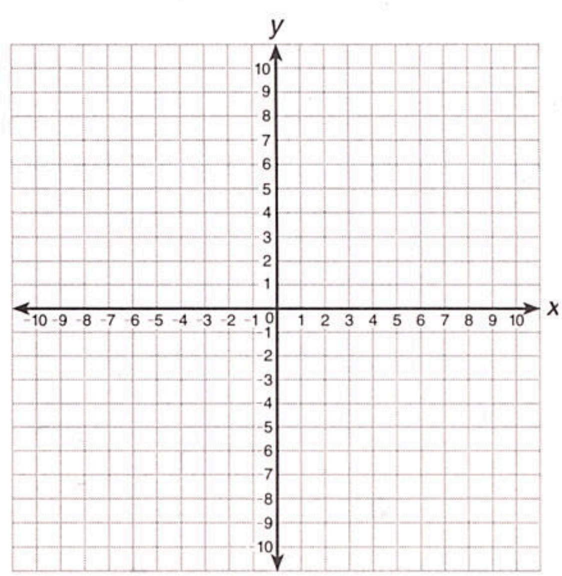
   radius __________



4. $(x - 6)^2 + (y + 7)^2 = 4$

   center __________

   radius __________

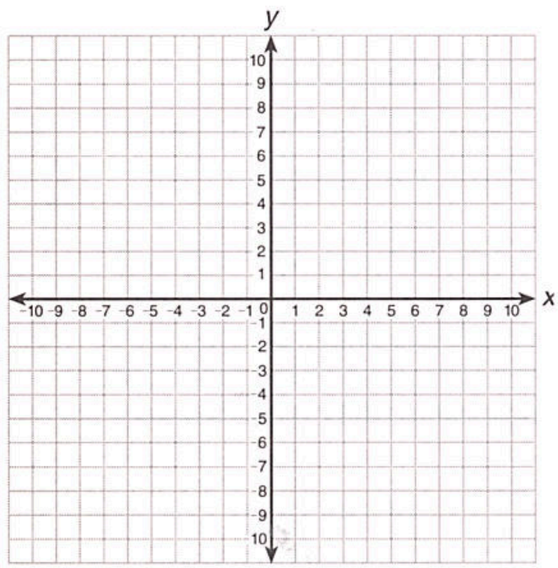


Benchmark Codes: MA.912.G.6.6, MA.912.G.6.7

Directions: For questions 5 through 8, write the equation of the circle given its center and radius. Then, graph the circle.

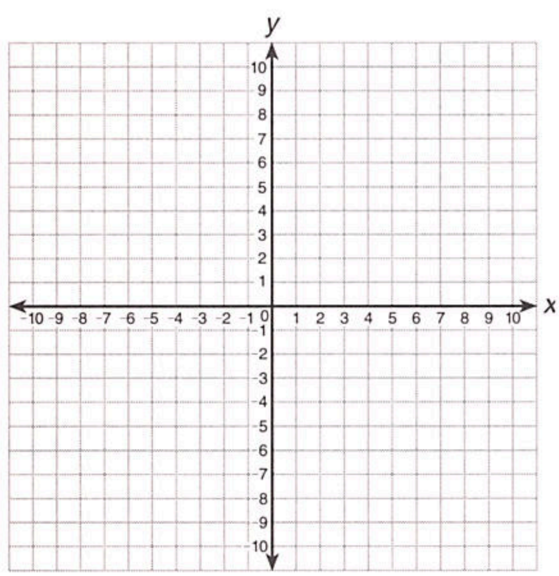
5. center = (2, 4); radius = 1

   equation: __________



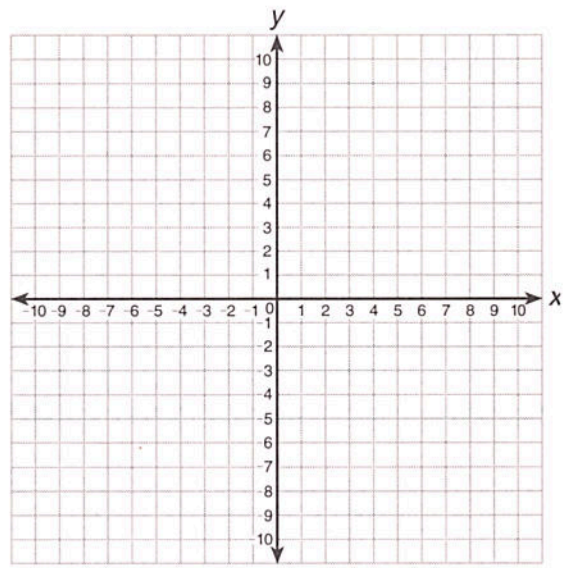
6. center = (0, 0); radius = 6

   equation: __________



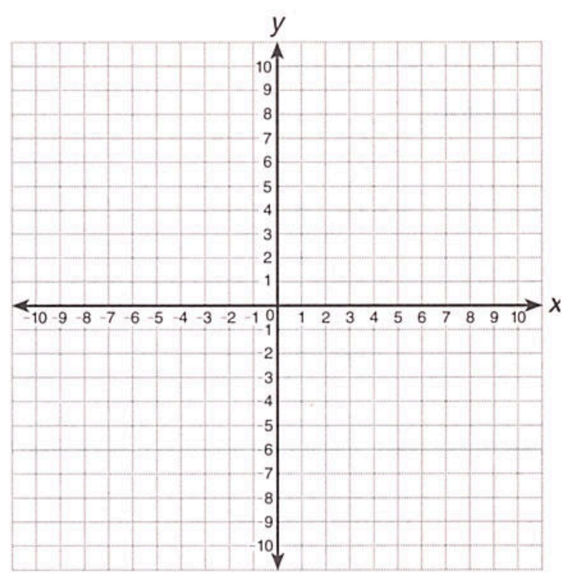
7. center = (3, 6); radius = 2

   equation: __________



8. center = (−1, −6); radius = 3

   equation: __________



Geometry Practice begins on the following page.