Lesson 9: Deductive Reasoning

Deductive reasoning is the process of arriving at a conclusion from premises that are given and accepted. This process is also called making an **argument**. In this lesson, you will analyze and make deductive arguments.

Syllogisms

A simple form of deductive reasoning, called a **syllogism**, involves combining a major premise and a minor premise to reach a conclusion.

- A **major premise** is a general statement about an entire group.
- A **minor premise** is a specific statement indicating that an individual case belongs to the group.
- A **conclusion** asserts that the general statement, which applies to the group, also applies to the individual case.

Example

- All polygons with exactly 4 sides are quadrilaterals. (major premise)
- A trapezoid has exactly four sides. (minor premise)
- Therefore, a trapezoid is a quadrilateral. (conclusion)

Practice

Directions: For questions 1 and 2, state the missing premise or conclusion.

1. major premise: All sides of an equilateral triangle are congruent.

   minor premise: In an equilateral triangle *ABC*, $\overline{AB}$ and $\overline{BC}$ are both sides.

   conclusion: ____.

2. major premise: Every right triangle has a right angle.

   minor premise: ____.

   conclusion: Therefore, $\triangle XYZ$ has one right angle.

Benchmark Codes: MA.912.D.6.3

Conditional Statements

A **conditional statement** is written in the form ***if p, then q***. If a given condition is met (*if p*), then another condition is true or an event will happen (*then q*). The if-clause is the **hypothesis**; the then-clause is the **conclusion**.

Example

If tomorrow is Thanksgiving, then today is Wednesday.
(hypothesis) (conclusion)

If a rectangle has four equal sides, then it is a square.
(hypothesis) (conclusion)

The **law of syllogisms** states that a conclusion can be drawn from two true conditional statements. Assuming *if p, then q* and *if q, then r* are true, then *if p, then r* is also true.

Example

If tomorrow is Thanksgiving, then today is Wednesday.	(conditional statement)
Tomorrow is Thanksgiving.	(minor premise)
Therefore, today is Wednesday.	(conclusion)

Practice

Directions: For questions 1 and 2, use the conditional statement and the minor premise to make a conclusion.

1. If a triangle has 3 equal angles, then it is equilateral.

   $\triangle XYZ$ has 3 equal angles.

   Therefore, ______________________________.

2. If two angle measures have a sum of 90°, then the angles are complementary.

   The measures of $\angle F$ and $\angle G$ have a sum of 90°.

   Therefore, ______________________________.

Critiquing for Validity

Deductive reasoning may be either valid or invalid. **Valid** reasoning arrives at conclusions that, based on the given premises and conditions, are absolutely true. **Invalid** reasoning arrives at conclusions that, based on the given premises and conditions, are false or not necessarily true. The process of evaluating deductive arguments for their validity is called **critiquing**.

Example

This is an example of valid reasoning:

If a 4-sided polygon has 4 right angles, then it is a rectangle. (true)

Polygon *ABCD* has four right angles. (assumed to be true)

Therefore, polygon *ABCD* is a rectangle. (true)

Could polygon *ABCD* also be a square? Yes, but remember that a square is a type of rectangle. The reasoning is valid.

Example

This is an example of invalid reasoning:

A person who lives in Florida lives in the United States. (true)

Mike lives in the United States. (assumed to be true)

Therefore, Mike lives in Florida. (not necessarily true)

The major premise is true, and the minor premise is assumed to be true. But do these true premises necessarily make the conclusion true? No. Mike could live in Florida—but he could also live in any of the other 49 states. The reasoning is invalid.

Practice

Directions: For questions 1 through 5, critique the argument for its validity. If the argument is invalid, state why.

1. All rock bands use instruments to play music.

   The Rats are a rock band.

   Therefore, The Rats use instruments to play music.

   ____

   ____

Benchmark Codes: MA.912.D.6.3, MA.912.D.6.4

2. Citizens of Florida must be at least 18 years old to vote.

   Cynthia does not vote.

   Therefore, Cynthia is not yet 18 years old.

   ____

   ____

3. If a quadrilateral is a square, then it is a parallelogram.

   Quadrilateral X is a parallelogram.

   Therefore, Quadrilateral X is a square.

   ____

   ____

4. Congruent quadrilaterals have identical angle measures.

   Quadrilaterals A and B are congruent.

   Therefore, quadrilaterals A and B have identical angle measures.

   ____

   ____

5. If two parallel lines are cut by a transversal, then each pair of corresponding angles is congruent.

   $\angle L$ and $\angle M$ are corresponding angles.

   Therefore, $\angle L$ and $\angle M$ are congruent.

   ____

   ____

Converse Statements

The **converse** of a conditional statement is formed by switching the places of the hypothesis and the conclusion. The sentence *if p, then q* becomes *if q, then p.*

Example

If a polygon has exactly five sides, then it is a pentagon. (given)

If a polygon is a pentagon, then it has exactly five sides. (converse)

Even if a given statement is true, the converse is not necessarily true.

Example

If two angles are both 30°, then they are congruent. (true)

If two angles are congruent, then they are both 30°. (false)

Likewise, a false original statement can produce a true converse.

Example

If you are exercising, then you are running on a treadmill. (false)

If you are running on a treadmill, then you are exercising. (true)

Benchmark Codes: MA.912.D.6.2, MA.912.D.6.3

Practice

Directions: For questions 1 through 5, form the converse of the given statement. Then state whether the statement and its converse are true or false.

1. If you live in Florida, then you live in the United States.

____

____

2. If a polygon is a square, then it has four right angles.

____

____

3. If a triangle is a right triangle, then it is a 30°–60°–90° triangle.

____

____

4. If the sides of one triangle are congruent to the sides of another triangle, then the two triangles are congruent.

____

____

5. If a number is an integer, then it is a whole number.

____

____

Inverse Statements

The **inverse** of a conditional statement is formed by negating the hypothesis and the conclusion. The sentence *if p, then q* becomes *if not p, then not q*. The inverse is sometimes written as *if ~p, then ~q*.

Example

If it is Wednesday, then Sara will go to school. (given)

If it is not Wednesday, then Sara will not go to school. (inverse)

As with converses, a true sentence can produce a false inverse, and a false sentence can produce a true inverse.

Example

If two lines are perpendicular, then they intersect. (true)

If two lines are not perpendicular, then they do not intersect. (false)

Practice

Directions: For questions 1 through 4, form the inverse of the given statement. Then state whether the statement and its inverse are true or false.

1. If a car starts, then the car has an engine.

____

2. If an integer is even, then it is twice another nonzero integer.

____

3. If two triangles are congruent, then they are similar.

____

4. If a polygon has four sides of equal length, then it is a square.

____

Benchmark Codes: MA.912.D.6.2, MA.912.D.6.3

Contrapositive Statements

The **contrapositive** of a conditional statement is formed in two steps:

Step 1: **Form the converse of the statement.**

Step 2: **Form the inverse of the converse.**

In other words, the sentence *if p, then q* becomes *if not q, then not p.* It is also written as *if* $\sim q$*, then* $\sim p$.

Example

If a polygon has exactly four sides, then it is a quadrilateral. (given)

If a polygon is a quadrilateral, then it has exactly four sides. (converse)

If a polygon is not a quadrilateral, then it does not have exactly four sides. (inverse of the converse = contrapositive)

You have seen that true statements can have false converses or inverses, and that false statements can have true inverses or converses. Contrapositives are different. The contrapositive of a true statement is *always* true; the contrapositive of a false statement is *always* false.

Practice

Directions: For questions 1 through 3, write the contrapositive of the given statement. Then state whether the statement and its contrapositive are true or false.

1. If two angles are straight angles, their angle measures are both 180°.

____

2. If two segments are radii of the same circle, then they are congruent.

____

3. If two angles are both 160°, then they are both acute.

____

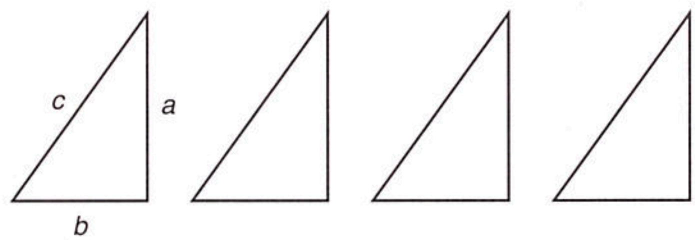
Geometric Deduction

Not every deductive argument is written with words. Mathematicians often use mathematical diagrams and terms to make arguments and derive conclusions.

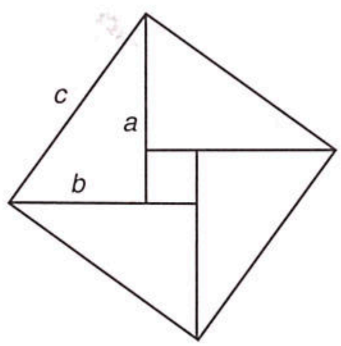
Example

Use deduction to derive the Pythagorean theorem ($a^2 + b^2 = c^2$).

Start with four copies of a right triangle of side lengths a, b, and c.



Arrange the triangles to make a square of side length c. At the center is a small square with side length $a - b$.



The area of the large square is c^2. This is equal to the sum of the areas of the four triangles and the small square.

area of four triangles	area of small square
$\left(\frac{1}{2}ab\right) \cdot 4$	$(a - b)^2$
$2ab$	$a^2 - 2ab + b^2$

area of four triangles + area of small square = area of large square

$$2ab + a^2 - 2ab + b^2 = c^2$$

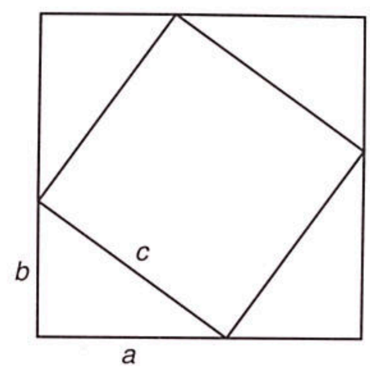
$$a^2 + b^2 = c^2$$

The key to this deductive process is finding two methods for calculating the area of the same figure.

Benchmark Codes: MA.912.G.5.1, MA.912.D.6.4, MA.912.G.8.4, MA.912.G.8.5

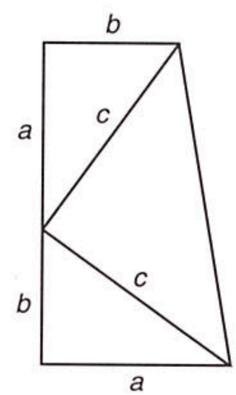
Practice

1. The four triangles from the previous example can be arranged in another way to make a square.



Derive the Pythagorean theorem by calculating the area of the large square in two ways. Show all of your work.

2. Recall that the formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$.



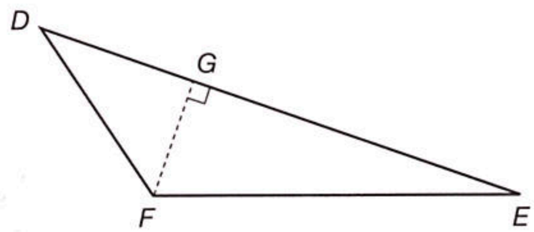
Derive the Pythagorean theorem by calculating the area of the trapezoid in two ways. Show all of your work.

Indirect Proofs

An **indirect proof** is a type of proof that shows a conclusion to be true by first assuming it is false and then proving the assumption to be invalid. An indirect proof is also called **proof by contradiction**.

Example

$\triangle DEF$ is a scalene triangle. Use an indirect proof to show that the altitude, $\overline{FG}$, will NOT bisect $\overline{DE}$.



To show that altitude $\overline{FG}$ will NOT bisect $\overline{DE}$ using an indirect proof, you must assume that $\overline{FG}$ DOES bisect $\overline{DE}$. Then show through a valid deductive argument that this assumption contradicts the given information.

Statement	Reason
1. $\triangle DEF$ is a scalene triangle.	1. Given
2. $\overline{FG}$ bisects $\overline{DE}$.	2. Assumption
3. G is the midpoint of $\overline{DE}$	3. A bisector intersects a segment at its midpoint.
4. $\overline{DG} \cong \overline{GE}$	4. A midpoint separates a segment into two congruent parts.
5. $\overline{FG} \perp \overline{DE}$	5. An altitude is perpendicular to the line it intersects
6. $m\angle DGF = 90°$	6. Perpendicular lines create 90° angles.
7. $m\angle EGF = 90°$	7. Perpendicular lines create 90° angles.
8. $m\angle DGF \cong m\angle EGF$	8. 90° = 90°.
9. $\overline{GF} \cong \overline{GF}$	9. The reflexive property of equality
10. $\triangle DGF \cong \triangle FGE$	10. SAS congruence
11. $\overline{DF} \cong \overline{EF}$	11. Corresponding parts of congruent triangles are congruent, contradiction of the fact that $\triangle DEF$ is a scalene triangle.

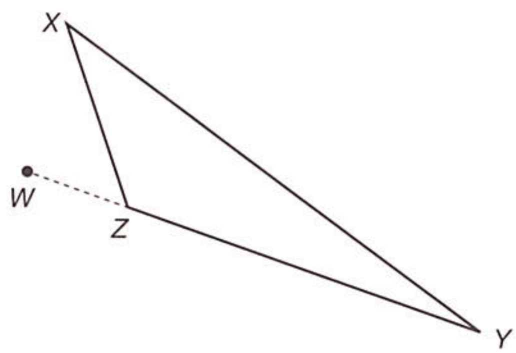
Because it is given that $\triangle DEF$ is a scalene triangle, the triangle has three sides with different lengths. Therefore, it cannot be proven that $\overline{DF} \cong \overline{EF}$. This forms a contradiction, which proves that altitude $\overline{FG}$ does NOT bisect $\overline{DE}$.

Benchmark Codes: MA.912.D.6.4, MA.912.G.8.5

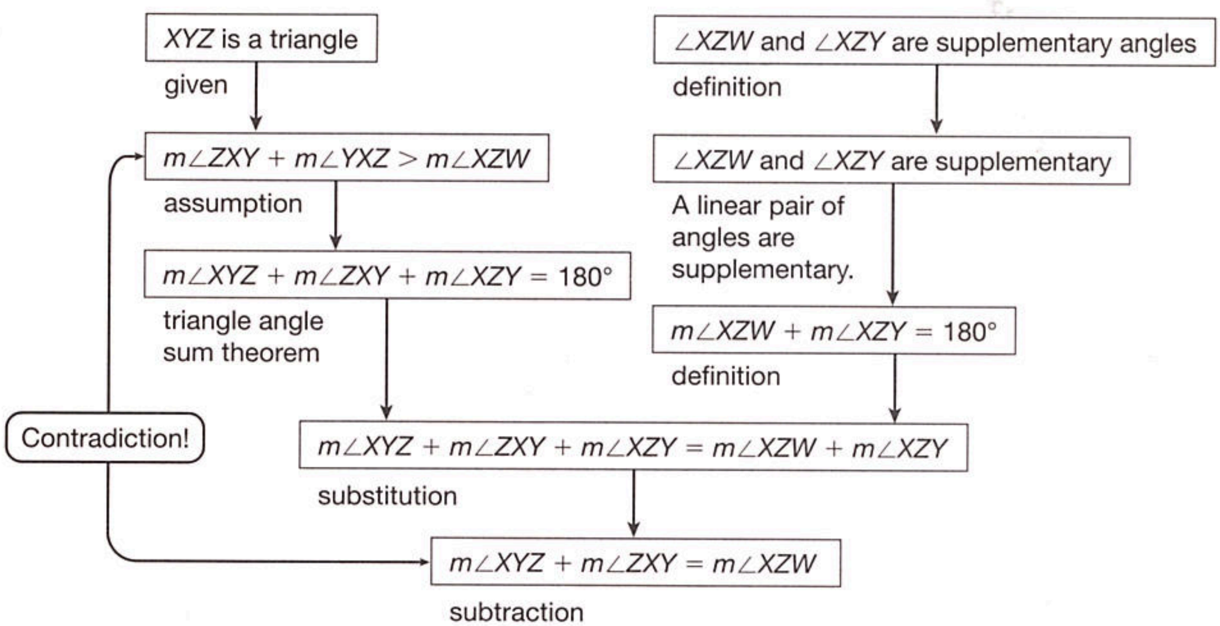
A **flow proof** is a type of proof that uses statements throughout the steps of an argument.

Example

The following figure shows $\triangle XYZ$ with an exterior angle drawn to point W. Use an indirect proof to show that $m\angle ZXY + m\angle ZYX$ is NOT greater than $m\angle XZW$.



To show that $m\angle ZXY + m\angle ZYX$ is NOT greater than $m\angle XZW$ using an indirect proof, you must assume that $m\angle ZXY + m\angle ZYX$ IS greater than $m\angle XZW$.



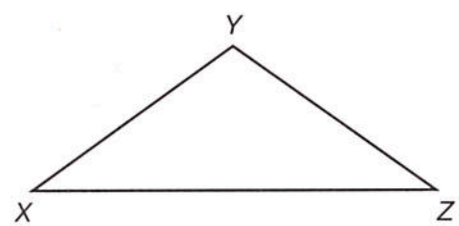
Because there is a contradiction, this proves that $m\angle ZXY + m\angle ZYX$ is NOT greater than $m\angle XZW$.

Practice

Directions: For questions 1 and 2, use a two-column indirect proof to prove the given statements.

1. Given that $\frac{x}{2}$ is a whole number, prove that x is a whole number.

2. Given that $\triangle XYZ$ is an isosceles triangle and $\angle Y$ is an obtuse angle, prove that $\angle X$ and $\angle Z$ are congruent.

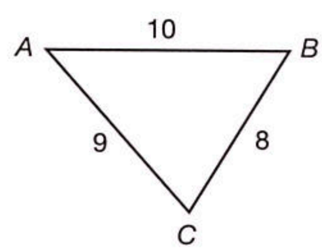


Benchmark Codes: MA.912.D.6.4, MA.912.G.8.5

Directions: For questions 3 and 4, use a flow indirect proof to prove the given statements.

3. Given that $-5x - 1 < 14$, prove that $x > -3$.

4. Given that $\triangle ABC$ is a scalene triangle with side lengths 8, 9, and 10, prove that $m\angle C > m\angle A$.



Geometry Practice begins on the following page.