Chapter 2

Polynomial and Rational Functions

2.8 Modeling Variation
Objectives:

- Solve direct variation problems
- Solve inverse variation problems
- Solve combined variation problems
- Solve problems involving joint variation.
Direct Variation

If a situation described by an equation in the form
\[ y = kx, \]
where \( k \) is a nonzero constant, we say that \( y \) varies directly as \( x \) or \( y \) is directly proportional to \( x \). The number \( k \) is called the constant of variation or the constant of proportionality.
Solving Variation Problems

1. Write an equation that models the given English statement.
2. Substitute the given pair of values into the equation in step 1 and find the value of $k$, the constant of variation.
3. Substitute the value of $k$ into the equation in step 1.
4. Use the equation from step 3 to answer the problem’s question.
Example: Solving a Direct Variation Problem

The number of gallons of water, \( W \), used when taking a shower varies directly as the time, \( t \), in minutes, in the shower. A shower lasting 5 minutes uses 30 gallons of water. How much water is used in a shower lasting 11 minutes?

Step 1 Write an equation.

\[
W = kt
\]

Step 2 Use the given values to find \( k \).

\[
W = kt \\
30 = k \cdot 5 \\
\frac{30}{5} = \frac{k \cdot 5}{5} \\
k = 6
\]
The number of gallons of water, \( W \), used when taking a shower varies directly as the time, \( t \), in minutes, in the shower. A shower lasting 5 minutes uses 30 gallons of water. How much water is used in a shower lasting 11 minutes?

**Step 3  Substitute the value of \( k \) into the equation.**

\[
W = kt
\]

\[
W = 6t
\]
The number of gallons of water, \( W \), used when taking a shower varies directly as the time, \( t \), in minutes, in the shower. A shower lasting 5 minutes uses 30 gallons of water. How much water is used in a shower lasting 11 minutes?

**Step 4  Answer the problem’s question.**

\[
W = 6t \quad W = 6(11) = 66
\]

A shower lasting 11 minutes will use 66 gallons of water.
Direct Variation with Powers

*y varies directly as the $n$th power of $x$* if there exists some nonzero constant $k$ such that

$$y = kx^n$$

We also say that *$y$ is directly proportional to the $n$th power of $x$.*
Example: Solving a Direct Variation Problem

The weight of a great white shark varies directly as the cube of its length. A great white shark caught off Catalina Island, California, was 15 feet long and weighed 2025 pounds. What was the weight of the 25-foot long shark in the novel Jaws?

Step 1 Write an equation.

\[ W = kl^3 \]
Example: Solving a Direct Variation Problem (continued)

The weight of a great white shark varies directly as the cube of its length. A great white shark caught off Catalina Island, California, was 15 feet long and weighed 2025 pounds. What was the weight of the 25-foot long shark in the novel *Jaws*?

**Step 2  Use the given values to find \( k \).**

\[
W = kl^3
\]

\[
2025 = k \cdot 15^3
\]

\[
2025 = k \cdot 3375
\]

\[
\frac{2025}{3375} = \frac{k \cdot 3375}{3375}
\]

\[
k = \frac{2025}{3375} = 0.6
\]
The weight of a great white shark varies directly as the cube of its length. A great white shark caught off Catalina Island, California, was 15 feet long and weighed 2025 pounds. What was the weight of the 25-foot long shark in the novel *Jaws*?

**Step 3** Substitute the value of $k$ into the equation.

$$W = kl^3$$

$$W = 0.6l^3$$
The weight of a great white shark varies directly as the cube of its length. A great white shark caught off Catalina Island, California, was 15 feet long and weighed 2025 pounds. What was the weight of the 25-foot long shark in the novel *Jaws*?

**Step 4  Answer the problem’s question.**

\[
W = 0.6l^3 \\
W = 0.6(25^3) \\
W = 0.6(15625) \\
W = 9375
\]

The weight of the 25-foot long shark in the novel *Jaws* was 9375 pounds.
Inverse Variation

If a situation is described by an equation in the form

\[ y = \frac{k}{x} \]

where \( k \) is a nonzero constant, we say that \( y \) varies inversely as \( x \) or \( y \) is inversely proportional to \( x \). The number \( k \) is called the constant of variation.
Example: Solving an Inverse Variation Problem

The length of a violin string varies inversely as the frequency of its vibrations. A violin string 8 inches long vibrates at a frequency of 640 cycles per second. What is the frequency of a 10-inch string?

Step 1 Write an equation.

\[ l = \frac{k}{f} \]
Example: Solving an Inverse Variation Problem (continued)

The length of a violin string varies inversely as the frequency of its vibrations. A violin string 8 inches long vibrates at a frequency of 640 cycles per second. What is the frequency of a 10-inch string?

Step 2  Use the given values to find $k$.

$$ l = \frac{k}{f} \quad 8 = \frac{k}{640} \quad 640 \times 8 = \frac{k}{640} \times 640 \quad k = 5120 $$
Example: Solving an Inverse Variation Problem (continued)

The length of a violin string varies inversely as the frequency of its vibrations. A violin string 8 inches long vibrates at a frequency of 640 cycles per second. What is the frequency of a 10-inch string?

Step 3  Substitute the value of $k$ into the equation.

$$l = \frac{k}{f} \quad l = \frac{5120}{f}$$
Example: Solving an Inverse Variation Problem (continued)

The length of a violin string varies inversely as the frequency of its vibrations. A violin string 8 inches long vibrates at a frequency of 640 cycles per second. What is the frequency of a 10-inch string?

Step 4 Answer the problem’s question.

\[ l = \frac{5120}{f} \quad \text{and} \quad 10 = \frac{5120}{f} \]  
\[ 10f = \frac{5120}{f} \quad \Rightarrow \quad 10f = 5120 \]

\[ f = \frac{5120}{10} = 512 \]

A violin string 10 inches long vibrates at a frequency of 512 cycles per second.
Combined Variation

In **combined variation**, direct variation and inverse variation occur at the same time.
Example: Solving a Combined Variation Problem

The number of minutes needed to solve an Exercise Set of variation problems varies directly as the number of problems and inversely as the number of people working to solve the problems. It takes 4 people 32 minutes to solve 16 problems. How many minutes will it take 8 people to solve 24 problems?

Step 1 Write an equation.

Let $m = \text{the number of minutes}$

$n = \text{the number of problems}$

$p = \text{the number of people}$

$$m = \frac{kn}{p}$$
Example: Solving a Combined Variation Problem (continued)

The number of minutes needed to solve an exercise set of variation problems varies directly as the number of problems and inversely as the number of people working to solve the problems. It takes 4 people 32 minutes to solve 16 problems. How many minutes will it take 8 people to solve 24 problems?

Step 2 Use the given values to find $k$.

$$m = \frac{kn}{p} \quad 32 = \frac{k \cdot 16}{4} \quad 32 = 4k \quad k = \frac{32}{4} = 8$$
The number of minutes needed to solve an exercise set of variation problems varies directly as the number of problems and inversely as the number of people working to solve the problems. It takes 4 people 32 minutes to solve 16 problems. How many minutes will it take 8 people to solve 24 problems?

**Step 3** Substitute the value of $k$ into the equation.

$$m = \frac{8n}{p}$$
Example: Solving a Combined Variation Problem (continued)

The number of minutes needed to solve an exercise set of variation problems varies directly as the number of problems and inversely as the number of people working to solve the problems. It takes 4 people 32 minutes to solve 16 problems. How many minutes will it take 8 people to solve 24 problems?

Step 4 Answer the problem’s question.

\[ m = \frac{8n}{p} \quad m = \frac{8(24)}{8} = 24 \]

It will take 8 people 24 minutes to solve 24 problems.
Joint Variation

Joint variation is a variation in which a variable varies directly as the product of two or more variables.

Thus, the equation $y = kxz$ is read “$y$ varies jointly as $x$ and $z$.”
Example: Solving a Joint Variation Problem

The volume of a cone, $V$, varies jointly as its height, $h$, and the square of its radius, $r$. A cone with a radius measuring 6 feet and a height measuring 10 feet has a volume of $120\pi$ cubic feet. Find the volume of a cone having a radius of 12 feet and a height of 2 feet.

Step 1 Write an equation.

$$V = khr^2$$
Example: Solving a Joint Variation Problem (continued)

The volume of a cone, \( V \), varies jointly as its height, \( h \), and the square of its radius, \( r \). A cone with a radius measuring 6 feet and a height measuring 10 feet has a volume of \( 120\pi \) cubic feet. Find the volume of a cone having a radius of 12 feet and a height of 2 feet.

Step 2  Use the given values to find \( k \).

\[
V = khr^2 \\
120\pi = k \cdot 10 \cdot 6^2 \\
120\pi = 360k
\]

\[
k = \frac{120\pi}{360} = \frac{\pi}{3}
\]
The volume of a cone, \( V \), varies jointly as its height, \( h \), and the square of its radius, \( r \). A cone with a radius measuring 6 feet and a height measuring 10 feet has a volume of \( 120\pi \) cubic feet. Find the volume of a cone having a radius of 12 feet and a height of 2 feet.

**Step 3** Substitute the value of \( k \) into the equation.

\[
V = \frac{\pi}{3} hr^2
\]
Example: Solving a Joint Variation Problem (continued)

The volume of a cone, $V$, varies jointly as its height, $h$, and the square of its radius, $r$. A cone with a radius measuring 6 feet and a height measuring 10 feet has a volume of $120\pi$ cubic feet. Find the volume of a cone having a radius of 12 feet and a height of 2 feet.

Step 4 Answer the problem’s question.

$$V = \frac{\pi}{3} hr^2$$

The volume of the cone is $96\pi$ cubic feet.

$$V = \frac{\pi}{3} \cdot 2 \cdot 12^2 = 96\pi$$