85. Follow the outline below and use mathematical induction to prove the Binomial Theorem:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

a. Verify the formula for $n = 1$.

b. Replace n with k and write the statement that is assumed true. Replace n with $k+1$ and write the statement that must be proved.

c. Multiply both sides of the statement assumed to be true by $a + b$. Add exponents on the left. On the right, distribute a and b, respectively.

d. Collect like terms on the right. At this point, you should have

$$(a+b)^{k+1} = \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^k b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2 + \left[\binom{k}{2} + \binom{k}{3}\right]a^{k-2}b^3 + \cdots + \left[\binom{k}{k-1} + \binom{k}{k}\right]ab^k + \binom{k}{k}b^{k+1}.$$

e. Use the result of Exercise 84 to add the binomial sums in brackets. For example, because $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$, then $\binom{k}{0} + \binom{k}{1} = \binom{k+1}{1}$ and $\binom{k}{1} + \binom{k}{2} = \binom{k+1}{2}$.

f. Because $\binom{k}{0} = \binom{k+1}{0}$ (why?) and $\binom{k}{k} = \binom{k+1}{k+1}$ (why?), substitute these results and the results from part (e) into the equation in part (d). This should give the statement that we were required to prove in the second step of the mathematical induction process.

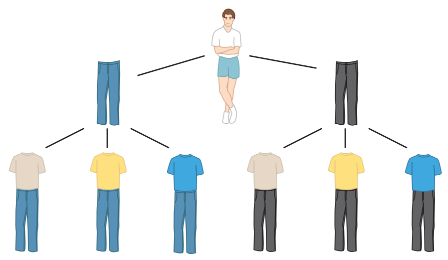
Preview Exercises

Exercises 86–88 will help you prepare for the material covered in the next section.

86. Evaluate $\dfrac{n!}{(n-r)!}$ for $n = 20$ and $r = 3$.

87. Evaluate $\dfrac{n!}{(n-r)!\,r!}$ for $n = 8$ and $r = 3$.

88. You can choose from two pairs of jeans (one blue, one black) and three T-shirts (one beige, one yellow, and one blue), as shown in the diagram.



True or false: The diagram shows that you can form 2×3, or 6, different outfits.

Section 11.6 Counting Principles, Permutations, and Combinations

Objectives

1. Use the Fundamental Counting Principle.
2. Use the permutations formula.
3. Distinguish between permutation problems and combination problems.
4. Use the combinations formula.

Have you ever imagined what your life would be like if you won the lottery? What changes would you make? Before you fantasize about becoming a person of leisure with a staff of obedient elves, think about this: The probability of winning top prize in the lottery is about the same as the probability of being struck by lightning. There are millions of possible number combinations in lottery games and only one way of winning the grand prize. Determining the probability of winning involves calculating the chance of getting the winning combination from all possible outcomes. In this section, we begin preparing for the surprising world of probability by looking at methods for counting possible outcomes.

1 Use the Fundamental Counting Principle.

The Fundamental Counting Principle

It's early morning, you're groggy, and you have to select something to wear for your 8 A.M. class. (What *were* you thinking of when you signed up for a class at that hour?!) Fortunately, your "lecture wardrobe" is rather limited—just two pairs of jeans to choose from (one blue, one black), three T-shirts to choose from (one beige, one yellow, and one blue), and two pairs of sneakers to select from (one black pair, one red pair). Your possible outfits are shown in **Figure 11.8**.

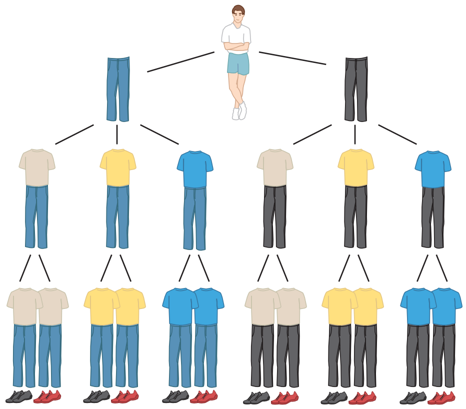


Figure 11.8 Selecting a wardrobe

The **tree diagram**, so named because of its branches, shows that you can form 12 outfits from your two pairs of jeans, three T-shirts, and two pairs of sneakers. Notice that the number of outfits can be obtained by multiplying the number of choices for jeans, 2, the number of choices for the T-shirts, 3, and the number of choices for the sneakers, 2:

$$2 \cdot 3 \cdot 2 = 12.$$

We can generalize this idea to any two or more groups of items—not just jeans, T-shirts, and sneakers—with the **Fundamental Counting Principle**:

The Fundamental Counting Principle

The number of ways in which a series of successive things can occur is found by multiplying the number of ways in which each thing can occur.

For example, if you own 30 pairs of jeans, 20 T-shirts, and 12 pairs of sneakers, you have

$$30 \cdot 20 \cdot 12 = 7200$$

choices for your wardrobe!

EXAMPLE 1 Options in Planning a Course Schedule

Next semester you are planning to take three courses—math, English, and humanities. Based on time blocks and highly recommended professors, there are 8 sections of math, 5 of English, and 4 of humanities that you find suitable. Assuming no scheduling conflicts, how many different three-course schedules are possible?

Solution This situation involves making choices with three groups of items.

Math	English	Humanities
8 choices	5 choices	4 choices

We use the Fundamental Counting Principle to find the number of three-course schedules. Multiply the number of choices for each of the three groups:

$$8 \cdot 5 \cdot 4 = 160.$$

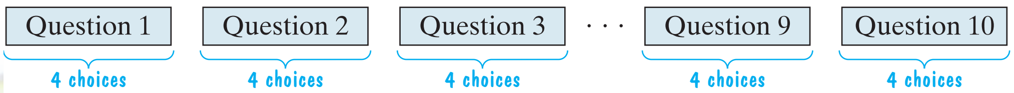
Thus, there are 160 different three-course schedules.

Check Point 1 A pizza can be ordered with three choices of size (small, medium, or large), four choices of crust (thin, thick, crispy, or regular), and six choices of toppings (ground beef, sausage, pepperoni, bacon, mushrooms, or onions). How many different one-topping pizzas can be ordered?

EXAMPLE 2 A Multiple-Choice Test

You are taking a multiple-choice test that has ten questions. Each of the questions has four answer choices, with one correct answer per question. If you select one of these four choices for each question and leave nothing blank, in how many ways can you answer the questions?

Solution This situation involves making choices with ten questions.



We use the Fundamental Counting Principle to determine the number of ways that you can answer the questions on the test. Multiply the number of choices, 4, for each of the ten questions.

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^{10} = 1{,}048{,}576$$

Thus, you can answer the questions in 1,048,576 different ways.

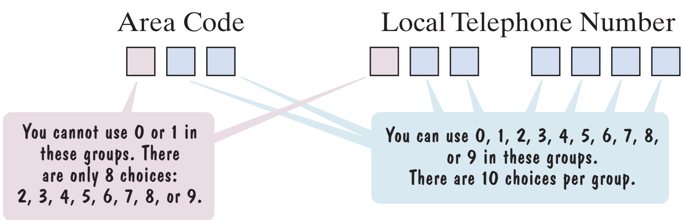
Are you surprised that there are over one million ways of answering a ten-question multiple-choice test? Of course, there is only one way to answer the test and receive a perfect score. The probability of guessing your way into a perfect score involves calculating the chance of getting a perfect score, just one way from all 1,048,576 possible outcomes. In short, prepare for the test and do not rely on guessing!

Check Point 2 You are taking a multiple-choice test that has six questions. Each of the questions has three answer choices, with one correct answer per question. If you select one of these three choices for each question and leave nothing blank, in how many ways can you answer the questions?

EXAMPLE 3 Telephone Numbers in the United States

Telephone numbers in the United States begin with three-digit area codes followed by seven-digit local telephone numbers. Area codes and local telephone numbers cannot begin with 0 or 1. How many different telephone numbers are possible?

Solution This situation involves making choices with ten groups of items.



The number of possible ways of playing the first four moves on each side in a game of chess is 318,979,564,000.

Running Out of Telephone Numbers

By the year 2020, portable telephones used for business and pleasure will all be videophones. At that time, U.S. population is expected to be 323 million. Faxes, beepers, cell phones, computer phone lines, and business lines may result in certain areas running out of phone numbers. Solution: Add more digits!

With or without extra digits, we expect that the 2020 videophone greeting will still be "hello," a word created by Thomas Edison in 1877. Phone inventor Alexander Graham Bell preferred "ahoy," but "hello" won out, appearing in the *Oxford English Dictionary* in 1883.

(Source: New York Times)

Here are the numbers of choices for each of the ten groups of items:

Area Code			Local Telephone Number						
8	10	10	8	10	10	10	10	10	10

We use the Fundamental Counting Principle to determine the number of different telephone numbers that are possible. The total number of telephone numbers possible is

$$8 \cdot 10 \cdot 10 \cdot 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6{,}400{,}000{,}000.$$

There are six billion four hundred million different telephone numbers that are possible.

Check Point 3 License plates in a particular state display two letters followed by three numbers, such as AT-887 or BB-013. How many different license plates can be manufactured?

2 Use the permutations formula.

Permutations

You are the coach of a little league baseball team. There are 13 players on the team (and lots of parents hovering in the background, dreaming of stardom for their little "Manny Ramirez"). You need to choose a batting order having 9 players. The order makes a difference, because, for instance, if bases are loaded and "Little Manny" is fourth or fifth at bat, his possible home run will drive in three additional runs. How many batting orders can you form?

You can choose any of 13 players for the first person at bat. Then you will have 12 players from which to choose the second batter, then 11 from which to choose the third batter, and so on. The situation can be shown as follows:

Batter 1	Batter 2	Batter 3	Batter 4	Batter 5	Batter 6	Batter 7	Batter 8	Batter 9
13 choices	12 choices	11 choices	10 choices	9 choices	8 choices	7 choices	6 choices	5 choices

We use the Fundamental Counting Principle to find the number of batting orders. The total number of batting orders is

$$13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 259{,}459{,}200.$$

Nearly 260 million batting orders are possible for your 13-player little league team. Each batting order is called a *permutation* of 13 players taken 9 at a time. The number of permutations of 13 players taken 9 at a time is 259,459,200.

A **permutation** is an ordered arrangement of items that occurs when

- No item is used more than once. (Each of the 9 players in the batting order bats exactly once.)
- The order of arrangement makes a difference.

We can obtain a formula for finding the number of permutations of 13 players taken 9 at a time by rewriting our computation:

$$13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \boxed{4 \cdot 3 \cdot 2 \cdot 1}}{\boxed{4 \cdot 3 \cdot 2 \cdot 1}} = \frac{13!}{4!} = \frac{13!}{(13-9)!}.$$

Thus, the number of permutations of 13 things taken 9 at a time is $\frac{13!}{(13-9)!}$. The special notation $_{13}P_9$ is used to replace the phrase "the number of permutations of 13 things taken 9 at a time." Using this new notation, we can write

$$_{13}P_9 = \frac{13!}{(13-9)!}.$$

The numerator of this expression is the number of items, 13 team members, expressed as a factorial: 13!. The denominator is also a factorial. It is the factorial of the difference between the number of items, 13, and the number of items in each permutation, 9 batters: $(13 - 9)!$.

The notation $_nP_r$ means the **number of permutations of n things taken r at a time**. We can generalize from the situation in which 9 batters were taken from 13 players. By generalizing, we obtain the following formula for the number of permutations if r items are taken from n items.

Study Tip

Because all permutation problems are also fundamental counting problems, they can be solved using the formula for $_nP_r$ or using the Fundamental Counting Principle.

Permutations of n Things Taken r at a Time

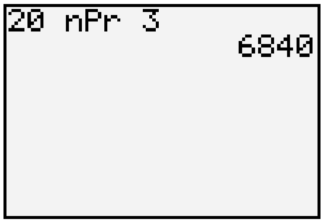
The number of possible permutations if r items are taken from n items is

$$_nP_r = \frac{n!}{(n-r)!}.$$

Technology

Graphing utilities have a menu item for calculating permutations, usually labeled $\boxed{_nP_r}$. For example, to find $_{20}P_3$, the keystrokes are

20 $\boxed{_nP_r}$ 3 $\boxed{\text{ENTER}}$.



If you are using a scientific calculator, check your manual for the location of the menu item for calculating permutations and the required keystrokes.

EXAMPLE 4 Using the Formula for Permutations

You and 19 of your friends have decided to form an Internet marketing consulting firm. The group needs to choose three officers—a CEO, an operating manager, and a treasurer. In how many ways can those offices be filled?

Solution Your group is choosing $r = 3$ officers from a group of $n = 20$ people (you and 19 friends). The order in which the officers are chosen matters because the CEO, the operating manager, and the treasurer each have different responsibilities. Thus, we are looking for the number of permutations of 20 things taken 3 at a time. We use the formula

$$_nP_r = \frac{n!}{(n-r)!}$$

with $n = 20$ and $r = 3$.

$$_{20}P_3 = \frac{20!}{(20-3)!} = \frac{20!}{17!} = \frac{20 \cdot 19 \cdot 18 \cdot 17!}{17!} = \frac{20 \cdot 19 \cdot 18 \cdot \cancel{17!}}{\cancel{17!}} = 20 \cdot 19 \cdot 18 = 6840$$

Thus, there are 6840 different ways of filling the three offices.

Check Point 4 A corporation has seven members on its board of directors. In how many different ways can it elect a president, vice-president, secretary, and treasurer?

How to Pass the Time for $2\frac{1}{2}$ Million Years

If you were to arrange 15 different books on a shelf and it took you one minute for each permutation, the entire task would take 2,487,965 years.

Source: Isaac Asimov's *Book of Facts.*

EXAMPLE 5 Using the Formula for Permutations

You need to arrange seven of your favorite books along a small shelf. How many different ways can you arrange the books, assuming that the order of the books makes a difference to you?

Solution Because you are using all seven of your books in every possible arrangement, you are arranging $r = 7$ books from a group of $n = 7$ books. Thus, we are looking for the number of permutations of 7 things taken 7 at a time. We use the formula

$$_nP_r = \frac{n!}{(n-r)!}$$

with $n = 7$ and $r = 7$.

$$_7P_7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 5040$$

Thus, you can arrange the books in 5040 ways. There are 5040 different possible permutations.

Check Point 5 In how many ways can six books be lined up along a shelf?

3 Distinguish between permutation problems and combination problems.

Combinations

Discussing the tragic death of actor Heath Ledger at age 28, *USA Today* (Jan. 30, 2008) cited five people who had achieved cult-figure status after death. Made iconic by death: Marilyn Monroe (actress, 1927–1962), James Dean (actor, 1931–1955), Jim Morrison (musician and lead singer of The Doors, 1943–1971), Janis Joplin (blues/rock singer, 1943–1970), and Jimi Hendrix (guitar virtuoso, 1943–1970).

Imagine that you ask your friends the following question: "Of these five people, which three would you select to be included in a documentary featuring the best of their work?" You are not asking your friends to rank their three favorite artists in any kind of order—they should merely select the three to be included in the documentary.

One friend answers, "Jim Morrison, Janis Joplin, and Jimi Hendrix." Another responds, "Jimi Hendrix, Janis Joplin, and Jim Morrison." These two people have the same artists in their group of selections, even if they are named in a different order. We are interested *in which artists are named, not the order in which they are named*, for the documentary. Because the items are taken without regard to order, this is not a permutation problem. No ranking of any sort is involved.

Later on, you ask your roommate which three artists she would select for the documentary. She names Marilyn Monroe, James Dean, and Jimi Hendrix. Her selection is different from those of your two other friends because different entertainers are cited.

Mathematicians describe the group of artists given by your roommate as a *combination*. A **combination** of items occurs when

- The items are selected from the same group (the five stars who were made iconic by death).
- No item is used more than once. (You may view Jimi Hendrix as a guitar god, but your three selections cannot be Jimi Hendrix, Jimi Hendrix, and Jimi Hendrix.)
- The order of the items makes no difference. (Morrison, Joplin, Hendrix is the same group in the documentary as Hendrix, Joplin, Morrison.)

Do you see the difference between a permutation and a combination? A permutation is an ordered arrangement of a given group of items. A combination is a group of items taken without regard to their order. **Permutation** problems involve situations in which **order matters**. **Combination** problems involve situations in which the **order** of the items **makes no difference**.

EXAMPLE 6 Distinguishing between Permutations and Combinations

For each of the following problems, determine whether the problem is one involving permutations or combinations. (It is not necessary to solve the problem.)

a. Six students are running for student government president, vice-president, and treasurer. The student with the greatest number of votes becomes the president, the second highest vote-getter becomes vice-president, and the student who gets the third largest number of votes will be treasurer. How many different outcomes are possible for these three positions?

b. Six people are on the board of supervisors for your neighborhood park. A three-person committee is needed to study the possibility of expanding the park. How many different committees could be formed from the six people?

c. Baskin-Robbins offers 31 different flavors of ice cream. One of their items is a bowl consisting of three scoops of ice cream, each a different flavor. How many such bowls are possible?

Solution

a. Students are choosing three student government officers from six candidates. The order in which the officers are chosen makes a difference because each of the offices (president, vice-president, treasurer) is different. Order matters. This is a problem involving permutations.

b. A three-person committee is to be formed from the six-person board of supervisors. The order in which the three people are selected does not matter because they are not filling different roles on the committee. Because order makes no difference, this is a problem involving combinations.

c. A three-scoop bowl of three different flavors is to be formed from Baskin-Robbins's 31 flavors. The order in which the three scoops of ice cream are put into the bowl is irrelevant. A bowl with chocolate, vanilla, and strawberry is exactly the same as a bowl with vanilla, strawberry, and chocolate. Different orderings do not change things, and so this is a problem involving combinations.

Check Point 6 For each of the following problems, explain if the problem is one involving permutations or combinations. (It is not necessary to solve the problem.)

a. How many ways can you select 6 free DVDs from a list of 200 DVDs?

b. In a race in which there are 50 runners and no ties, in how many ways can the first three finishers come in?

4 Use the combinations formula.

A Formula for Combinations

We have seen that the notation ${}_nP_r$ means the number of permutations of n things taken r at a time. Similarly, the notation ${}_nC_r$ **means the number of combinations of n things taken r at a time**.

We can develop a formula for ${}_nC_r$ by comparing permutations and combinations. Consider the letters A, B, C, and D. The number of permutations of these four letters taken three at a time is

$$ {}_4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{4\cdot 3\cdot 2\cdot 1}{1} = 24. $$

Here are the 24 permutations:

ABC,	ABD,	ACD,	BCD,
ACB,	ADB,	ADC,	BDC,
BAC,	BAD,	CAD,	CBD,
BCA,	BDA,	CDA,	CDB,
CAB,	DAB,	DAC,	DBC,
CBA,	DBA,	DCA,	DCB.
This column contains only one combination, ABC.	This column contains only one combination, ABD.	This column contains only one combination, ACD.	This column contains only one combination, BCD.

Because the order of items makes no difference in determining combinations, each column of six permutations represents one combination. There is a total of four combinations:

ABC, ABD, ACD, BCD.

Thus, ${}_4C_3 = 4$: The number of combinations of 4 things taken 3 at a time is 4. With 24 permutations and only four combinations, there are 6, or 3!, times as many permutations as there are combinations.

In general, there are $r!$ times as many permutations of n things taken r at a time as there are combinations of n things taken r at a time. Thus, we find the number of combinations of n things taken r at a time by dividing the number of permutations of n things taken r at a time by $r!$.

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!\,r!} $$

Study Tip

The number of combinations if r items are taken from n items cannot be found using the Fundamental Counting Principle and requires the use of the formula shown on the right.

Combinations of n Things Taken r at a Time

The number of possible combinations if r items are taken from n items is

$$ {}_nC_r = \frac{n!}{(n-r)!\,r!}. $$

Notice that the formula for ${}_nC_r$ is the same as the formula for the binomial coefficient $\binom{n}{r}$.

EXAMPLE 7 Using the Formula for Combinations

A three-person committee is needed to study ways of improving public transportation. How many committees could be formed from the eight people on the board of supervisors?

Solution The order in which the three people are selected does not matter. This is a problem of selecting $r = 3$ people from a group of $n = 8$ people. We are

Technology

Graphing utilities have a menu item for calculating combinations, usually labeled $_nC_r$. For example, to find $_8C_3$, the keystrokes on most graphing utilities are

$$8 \; \boxed{_nC_r} \; 3 \; \boxed{\text{ENTER}}.$$

If you are using a scientific calculator, check your manual to see whether there is a menu item for calculating combinations.

If you use your calculator's factorial key to find $\frac{8!}{5!3!}$, be sure to enclose the factorials in the denominator with parentheses

$$8 \; \boxed{!} \; \boxed{\div} \; \boxed{(} \; 5 \; \boxed{!} \; \boxed{\times} \; 3 \; \boxed{!} \; \boxed{)}$$

pressing $\boxed{=}$ or $\boxed{\text{ENTER}}$ to obtain the answer.

looking for the number of combinations of eight things taken three at a time. We use the formula

$$_nC_r = \frac{n!}{(n-r)!r!}$$

with $n = 8$ and $r = 3$.

$$_8C_3 = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!} \cdot 3 \cdot 2 \cdot 1} = 56$$

Thus, 56 committees of three people each can be formed from the eight people on the board of supervisors.

Check Point 7 From a group of 10 physicians, in how many ways can four people be selected to attend a conference on acupuncture?

EXAMPLE 8 Using the Formula for Combinations

In poker, a person is dealt 5 cards from a standard 52-card deck. The order in which you are dealt the 5 cards does not matter. How many different 5-card poker hands are possible?

Solution Because the order in which the 5 cards are dealt does not matter, this is a problem involving combinations. We are looking for the number of combinations of $n = 52$ cards dealt $r = 5$ at a time. We use the formula

$$_nC_r = \frac{n!}{(n-r)!r!}$$

with $n = 52$ and $r = 5$.

$$_{52}C_5 = \frac{52!}{(52-5)!5!} = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!} \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$

Thus, there are 2,598,960 different 5-card poker hands possible. It surprises many people that more than 2.5 million 5-card hands can be dealt from a mere 52 cards.

If you are a card player, it does not get any better than to be dealt the 5-card poker hand shown in **Figure 11.9**. This hand is called a *royal flush*. It consists of an ace, king, queen, jack, and 10, all of the same suit: all hearts, all diamonds, all clubs, or all spades. The probability of being dealt a royal flush involves calculating the number of ways of being dealt such a hand: just 4 of all 2,598,960 possible hands. In the next section, we move from counting possibilities to computing probabilities.

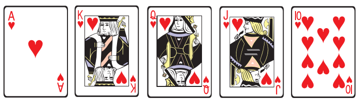


Figure 11.9 A royal flush

Check Point 8 How many different 4-card hands can be dealt from a deck that has 16 different cards?

Exercise Set 11.6

Practice Exercises

In Exercises 1–8, use the formula for $_nP_r$ to evaluate each expression.

1. $_9P_4$ **2.** $_7P_3$

3. $_8P_5$ **4.** $_{10}P_4$

5. $_6P_6$ **6.** $_9P_9$

7. $_8P_0$ **8.** $_6P_0$

In Exercises 9–16, use the formula for $_nC_r$ to evaluate each expression.

9. $_9C_5$ **10.** $_{10}C_6$

11. $_{11}C_4$ **12.** $_{12}C_5$

13. $_7C_7$ **14.** $_4C_4$

15. $_5C_0$ **16.** $_6C_0$

In Exercises 17–20, does the problem involve permutations or combinations? Explain your answer. (It is not necessary to solve the problem.)

17. A medical researcher needs 6 people to test the effectiveness of an experimental drug. If 13 people have volunteered for the test, in how many ways can 6 people be selected?

18. Fifty people purchase raffle tickets. Three winning tickets are selected at random. If first prize is \$1000, second prize is \$500, and third prize is \$100, in how many different ways can the prizes be awarded?

19. How many different four-letter passwords can be formed from the letters A, B, C, D, E, F, and G if no repetition of letters is allowed?

20. Fifty people purchase raffle tickets. Three winning tickets are selected at random. If each prize is \$500, in how many different ways can the prizes be awarded?

Practice Plus

In Exercises 21–28, evaluate each expression.

21. $\frac{{}_7P_3}{3!} - {}_7C_3$ **22.** $\frac{{}_{20}P_2}{2!} - {}_{20}C_2$ **23.** $1 - \frac{{}_3P_2}{{}_4P_3}$

24. $1 - \frac{{}_5P_3}{{}_{10}P_4}$ **25.** $\frac{{}_7C_3}{{}_5C_4} - \frac{98!}{96!}$ **26.** $\frac{{}_{10}C_3}{{}_6C_4} - \frac{46!}{44!}$

27. $\frac{{}_4C_2 \cdot {}_6C_1}{{}_{18}C_3}$ **28.** $\frac{{}_5C_1 \cdot {}_7C_2}{{}_{12}C_3}$

Application Exercises

Use the Fundamental Counting Principle to solve Exercises 29–40.

29. The model of the car you are thinking of buying is available in nine different colors and three different styles (hatchback, sedan, or station wagon). In how many ways can you order the car?

30. A popular brand of pen is available in three colors (red, green, or blue) and four writing tips (bold, medium, fine, or micro). How many different choices of pens do you have with this brand?

31. An ice cream store sells two drinks (sodas or milk shakes), in four sizes (small, medium, large, or jumbo), and five flavors (vanilla, strawberry, chocolate, coffee, or pistachio). In how many ways can a customer order a drink?

32. A restaurant offers the following lunch menu.

Main Course	Vegetables	Beverages	Desserts
Ham	Potatoes	Coffee	Cake
Chicken	Peas	Tea	Pie
Fish	Green beans	Milk	Ice cream
Beef		Soda	

If one item is selected from each of the four groups, in how many ways can a meal be ordered? Describe two such orders.

33. You are taking a multiple-choice test that has five questions. Each of the questions has three answer choices, with one correct answer per question. If you select one of these three choices for each question and leave nothing blank, in how many ways can you answer the questions?

34. You are taking a multiple-choice test that has eight questions. Each of the questions has three answer choices, with one correct answer per question. If you select one of these three choices for each question and leave nothing blank, in how many ways can you answer the questions?

35. In the original plan for area codes in 1945, the first digit could be any number from 2 through 9, the second digit was either 0 or 1, and the third digit could be any number except 0. With this plan, how many different area codes were possible?

36. How many different four-letter radio station call letters can be formed if the first letter must be W or K?

37. Six performers are to present their comedy acts on a weekend evening at a comedy club. One of the performers insists on being the last stand-up comic of the evening. If this performer's request is granted, how many different ways are there to schedule the appearances?

38. Five singers are to perform at a night club. One of the singers insists on being the last performer of the evening. If this singer's request is granted, how many different ways are there to schedule the appearances?

39. In the *Cambridge Encyclopedia of Language* (Cambridge University Press, 1987), author David Crystal presents five sentences that make a reasonable paragraph regardless of their order. The sentences are as follows:

- Mark had told him about the foxes.
- John looked out the window.
- Could it be a fox?
- However, nobody had seen one for months.
- He thought he saw a shape in the bushes.

How many different five-sentence paragraphs can be formed if the paragraph begins with "He thought he saw a shape in the bushes" and ends with "John looked out of the window"?

40. A television programmer is arranging the order that five movies will be seen between the hours of 6 P.M. and 4 A.M. Two of the movies have a G rating and they are to be shown in the first two time blocks. One of the movies is rated NC-17 and it is to be shown in the last of the time blocks, from 2 A.M. until 4 A.M. Given these restrictions, in how many ways can the five movies be arranged during the indicated time blocks?

Use the formula for ${}_nP_r$ to solve Exercises 41–48.

41. A club with ten members is to choose three officers—president, vice-president, and secretary-treasurer. If each office is to be held by one person and no person can hold more than one office, in how many ways can those offices be filled?

42. A corporation has ten members on its board of directors. In how many different ways can it elect a president, vice-president, secretary, and treasurer?

43. For a segment of a radio show, a disc jockey can play 7 songs. If there are 13 songs to select from, in how many ways can the program for this segment be arranged?

44. Suppose you are asked to list, in order of preference, the three best movies you have seen this year. If you saw 20 movies during the year, in how many ways can the three best be chosen and ranked?

45. In a race in which six automobiles are entered and there are no ties, in how many ways can the first three finishers come in?

46. In a production of *West Side Story*, eight actors are considered for the male roles of Tony, Riff, and Bernardo. In how many ways can the director cast the male roles?

47. Nine bands have volunteered to perform at a benefit concert, but there is only enough time for five of the bands to play. How many lineups are possible?

48. How many arrangements can be made using four of the letters of the word COMBINE if no letter is to be used more than once?

Use the formula for ${}_nC_r$ *to solve Exercises 49–56.*

49. An election ballot asks voters to select three city commissioners from a group of six candidates. In how many ways can this be done?

50. A four-person committee is to be elected from an organization's membership of 11 people. How many different committees are possible?

51. Of 12 possible books, you plan to take 4 with you on vacation. How many different collections of 4 books can you take?

52. There are 14 standbys who hope to get seats on a flight, but only 6 seats are available on the plane. How many different ways can the 6 people be selected?

53. You volunteer to help drive children at a charity event to the zoo, but you can fit only 8 of the 17 children present in your van. How many different groups of 8 children can you drive?

54. Of the 100 people in the U.S. Senate, 18 serve on the Foreign Relations Committee. How many ways are there to select Senate members for this committee (assuming party affiliation is not a factor in selection)?

55. To win at LOTTO in the state of Florida, one must correctly select 6 numbers from a collection of 53 numbers (1 through 53). The order in which the selection is made does not matter. How many different selections are possible?

56. To win in the New York State lottery, one must correctly select 6 numbers from 59 numbers. The order in which the selection is made does not matter. How many different selections are possible?

In Exercises 57–66, solve by the method of your choice.

57. In a race in which six automobiles are entered and there are no ties, in how many ways can the first four finishers come in?

58. A book club offers a choice of 8 books from a list of 40. In how many ways can a member make a selection?

59. A medical researcher needs 6 people to test the effectiveness of an experimental drug. If 13 people have volunteered for the test, in how many ways can 6 people be selected?

60. Fifty people purchase raffle tickets. Three winning tickets are selected at random. If first prize is \$1000, second prize is \$500, and third prize is \$100, in how many different ways can the prizes be awarded?

61. From a club of 20 people, in how many ways can a group of three members be selected to attend a conference?

62. Fifty people purchase raffle tickets. Three winning tickets are selected at random. If each prize is \$500, in how many different ways can the prizes be awarded?

63. How many different four-letter passwords can be formed from the letters A, B, C, D, E, F, and G if no repetition of letters is allowed?

64. Nine comedy acts will perform over two evenings. Five of the acts will perform on the first evening and the order in which the acts perform is important. How many ways can the schedule for the first evening be made?

65. Using 15 flavors of ice cream, how many cones with three different flavors can you create if it is important to you which flavor goes on the top, middle, and bottom?

66. Baskin-Robbins offers 31 different flavors of ice cream. One of their items is a bowl consisting of three scoops of ice cream, each a different flavor. How many such bowls are possible?

Exercises 67–72 are based on the following jokes about books:

- *"Outside of a dog, a book is man's best friend. Inside of a dog, it's too dark to read."—Groucho Marx*
- *"I recently bought a book of free verse. For \$12." —George Carlin*
- *"If a word in the dictionary was misspelled, how would we know?"—Steven Wright*
- *"Encyclopedia is a Latin term. It means 'to paraphrase a term paper.' "—Greg Ray*
- *"A bookstore is one of the only pieces of evidence we have that people are still thinking."—Jerry Seinfeld*
- *"I honestly believe there is absolutely nothing like going to bed with a good book. Or a friend who's read one." —Phyllis Diller*

67. In how many ways can these six jokes be ranked from best to worst?

68. If Phyllis Diller's joke about books is excluded, in how many ways can the remaining five jokes be ranked from best to worst?

69. In how many ways can people select their three favorite jokes from these comments about books?

70. In how many ways can people select their two favorite jokes from these comments about books?

71. If the order in which these jokes are told makes a difference in terms of how they are received, how many ways can they be delivered if George Carlin's joke is delivered first and Jerry Seinfeld's joke is told last?

72. If the order in which these jokes are told makes a difference in terms of how they are received, how many ways can they be delivered if a joke by a man is told first?

Writing in Mathematics

73. Explain the Fundamental Counting Principle.

74. Write an original problem that can be solved using the Fundamental Counting Principle. Then solve the problem.

75. What is a permutation?

76. Describe what ${}_nP_r$ represents.

77. Write a word problem that can be solved by evaluating ${}_7P_3$.

78. What is a combination?

79. Explain how to distinguish between permutation and combination problems.

80. Write a word problem that can be solved by evaluating ${}_7C_3$.

Technology Exercises

81. Use a graphing utility with an $\boxed{{}_nP_r}$ menu item to verify your answers in Exercises 1–8.

82. Use a graphing utility with an $\boxed{{}_nC_r}$ menu item to verify your answers in Exercises 9–16.

Critical Thinking Exercises

Make Sense? *In Exercises 83–86, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

83. I used the Fundamental Counting Principle to determine the number of five-digit ZIP codes that are available to the U.S. Postal Service.

84. I used the permutations formula to determine the number of ways the manager of a baseball team can form a 9-player batting order from a team of 25 players.

85. I used the combinations formula to determine how many different four-note sound sequences can be created from the notes C, D, E, F, G, A, and B.

86. I used the permutations formula to determine the number of ways people can select their 9 favorite baseball players from a team of 25 players.

In Exercises 87–90, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

87. The number of ways to choose four questions out of ten questions on an essay test is ${}_{10}P_4$.

88. If $r > 1$, ${}_nP_r$ is less than ${}_nC_r$.

89. ${}_7P_3 = 3!\,{}_7C_3$

90. The number of ways to pick a winner and first runner-up in a talent contest with 20 contestants is ${}_{20}C_2$.

91. Five men and five women line up at a checkout counter in a store. In how many ways can they line up if the first person in line is a woman and the people in line alternate woman, man, woman, man, and so on?

92. How many four-digit odd numbers less than 6000 can be formed using the digits 2, 4, 6, 7, 8, and 9?

93. A mathematics exam consists of 10 multiple-choice questions and 5 open-ended problems in which all work must be shown. If an examinee must answer 8 of the multiple-choice questions and 3 of the open-ended problems, in how many ways can the questions and problems be chosen?

Group Exercise

94. The group should select real-world situations where the Fundamental Counting Principle can be applied. These could involve the number of possible student ID numbers on your campus, the number of possible phone numbers in your community, the number of meal options at a local restaurant, the number of ways a person in the group can select outfits for class, the number of ways a condominium can be purchased in a nearby community, and so on. Once situations have been selected, group members should determine in how many ways each part of the task can be done. Group members will need to obtain menus, find out about telephone-digit requirements in the community, count shirts, pants, shoes in closets, visit condominium sales offices, and so on. Once the group reassembles, apply the Fundamental Counting Principle to determine the number of available options in each situation. Because these numbers may be quite large, use a calculator.

Preview Exercises

Exercises 95–97 will help you prepare for the material covered in the next section.

The figure shows that when a die is rolled, there are six equally likely outcomes: 1, 2, 3, 4, 5, or 6. Use this information to solve each exercise.

95. What fraction of the outcomes are less than 5?

96. What fraction of the outcomes are not less than 5?

97. What fraction of the outcomes are even or greater than 3?