6.2: Transforming and Combining Random Variables
After this section, you should be able to...

✓ DESCRIBE the effect of performing a linear transformation on a random variable

✓ COMBINE random variables and CALCULATE the resulting mean and standard deviation

✓ CALCULATE and INTERPRET probabilities involving combinations of Normal random variables
Linear Transformations on Random Variables
Review: Linear Transformations

In Chapter 2, we studied the effects of linear transformations on the shape, center, and spread of a distribution of data. Remember:

1. *Adding (or subtracting) a constant, $a$, to each observation:*  
   - Adds $a$ to measures of center and location.  
   - Does not change the shape or measures of spread.

2. *Multiplying (or dividing) each observation by a constant, $b$:*
   - Multiplies (divides) measures of center and location by $b$.  
   - Multiplies (divides) measures of spread by $|b|$.  
   - Does not change the shape of the distribution.
Linear Transformations on Random Variables

Multiplying (or dividing) each value of a random variable by a number $b$:

• Multiplies (divides) measures of center and location (mean, median, quartiles, percentiles) by $b$.

• Multiplies (divides) measures of spread (range, IQR, standard deviation) by $|b|$.

• Does not change the shape of the distribution.

**Note:** Multiplying a random variable by a constant $b$ multiplies the variance by $b^2$. 
Linear Transformations

Pete’s Jeep Tours offers a popular half-day trip in a tourist area. There must be at least 2 passengers for the trip to run, and the vehicle will hold up to 6 passengers. Define $X$ as the number of passengers on a randomly selected day.

<table>
<thead>
<tr>
<th>Passengers $x_i$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability $p_i$</td>
<td>0.15</td>
<td>0.25</td>
<td>0.35</td>
<td>0.20</td>
<td>0.05</td>
</tr>
</tbody>
</table>

The mean of $X$ is 3.75 and the standard deviation is 1.090.
Linear Transformations

Pete charges $150 per passenger. The random variable $C$ describes the amount Pete collects on a randomly selected day.

<table>
<thead>
<tr>
<th>Collected $c_i$</th>
<th>300</th>
<th>450</th>
<th>600</th>
<th>750</th>
<th>900</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability $p_i$</td>
<td>0.15</td>
<td>0.25</td>
<td>0.35</td>
<td>0.20</td>
<td>0.05</td>
</tr>
</tbody>
</table>

The mean of $C$ is $562.50 and the standard deviation is $163.50.
Compare the shape, center and spread of each distribution.
Linear Transformations on Random Variables

Adding the same number $a$ (which could be negative) to each value of a random variable:

• Adds $a$ to measures of center and location (mean, median, quartiles, percentiles).

• Does not change measures of spread (range, $IQR$, standard deviation).

• Does not change the shape of the distribution.
Linear Transformations

Consider Pete’s Jeep Tours again. We defined $C$ as the amount of money Pete collects on a randomly selected day.

<table>
<thead>
<tr>
<th>Collected $c_i$</th>
<th>300</th>
<th>450</th>
<th>600</th>
<th>750</th>
<th>900</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability $p_i$</td>
<td>0.15</td>
<td>0.25</td>
<td>0.35</td>
<td>0.20</td>
<td>0.05</td>
</tr>
</tbody>
</table>

The mean of $C$ is $562.50 and the standard deviation is $163.50.

It costs Pete $100 per trip to buy permits, gas, and a ferry pass. The random variable $V$ describes the profit Pete makes on a randomly selected day.

<table>
<thead>
<tr>
<th>Profit $v_i$</th>
<th>200</th>
<th>350</th>
<th>500</th>
<th>650</th>
<th>800</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability $p_i$</td>
<td>0.15</td>
<td>0.25</td>
<td>0.35</td>
<td>0.20</td>
<td>0.05</td>
</tr>
</tbody>
</table>

The mean of $V$ is $462.50 and the standard deviation is $163.50.

Compare the shape, center, and spread of the two probability distributions.
Bottom Line:

Whether we are dealing with data or random variables, the effects of a linear transformation are the same!!!
Combining Random Variables
Combining Random Variables

Before we can combine random variables, a determination about the independence of each variable from the other must be made. Probability models often assume independence when the random variables describe outcomes that appear unrelated to each other.

You should always ask yourself whether the assumption of independence seems reasonable.
Combining Random Variables

Let \( D \) = the number of passengers on a randomly selected Delta flight to Atlanta

Let \( A \) = the number of passengers on a randomly selected trip American Airlines flight to Atlanta

Define \( T = X + Y \). Calculate the mean and standard deviation of \( T \).

<table>
<thead>
<tr>
<th>Passengers ( x_i )</th>
<th>75</th>
<th>76</th>
<th>77</th>
<th>78</th>
<th>79</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability ( p_i )</td>
<td>0.15</td>
<td>0.25</td>
<td>0.35</td>
<td>0.20</td>
<td>0.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Passengers ( y_i )</th>
<th>75</th>
<th>76</th>
<th>77</th>
<th>78</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability ( p_i )</td>
<td>0.3</td>
<td>0.4</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Combining Random Variables

Let \( D \) = the number of passengers on a randomly selected Delta flight to Atlanta

Let \( A \) = the number of passengers on a randomly selected trip American Airlines flight to Atlanta

Define \( T = X + Y \). Calculate the mean and standard deviation of \( T \).

\[
\begin{array}{c|c|c|c|c|c}
\text{Passengers } x_i & 75 & 76 & 77 & 78 & 79 \\
\hline
\text{Probability } p_i & 0.15 & 0.25 & 0.35 & 0.20 & 0.05 \\
\end{array}
\]

Mean \( \mu_D = 76.75 \)  Standard Deviation \( \sigma_D = 1.0897 \)

\[
\begin{array}{c|c|c|c}
\text{Passengers } y_i & 75 & 76 & 77 & 78 \\
\hline
\text{Probability } p_i & 0.3 & 0.4 & 0.2 & 0.1 \\
\end{array}
\]

Mean \( \mu_A = 76.1 \)  Standard Deviation \( \sigma_A = 0.943 \)
Combining Random Variables: Mean

How many total passengers fly to Atlanta on a randomly selected day?

Delta: \( \mu_D = 76.75 \)
American: \( \mu_A = 76.10 \)
Total: \( 76.75 + 76.10 = 152.85 \) passengers to Atlanta daily

For any two random variables \( X \) and \( Y \), if \( T = X + Y \), then the expected value of \( T \) is

\[
E(T) = \mu_T = \mu_X + \mu_Y
\]

In general, the mean of the sum of several random variables is the sum of their means.
Combining Random Variables: Variance

How much variability is there in the total number of passengers who fly to Atlanta on a randomly selected day? (Hint: find the combined variance)

**Delta:** Mean $\mu_D = 76.75$  Standard Deviation $\sigma_D = 1.0897$

**American:** Mean $\mu_A = 76.1$  Standard Deviation $\sigma_A = 0.943$

**REMEMBER:** Standard Deviations do not add!!!
Combining Random Variables: Variance

Delta = (1.090)^2
American = (0.943)^2
Total Variance = (1.090)^2 + (0.943)^2 = 2.077

For any two independent random variables X and Y, if $T = X + Y$, then the variance of $T$ is

$$\sigma_T^2 = \sigma_X^2 + \sigma_Y^2$$

In general, the variance of the sum of several independent random variables is the sum of their variances.
Subtracting Random Variables: Mean

Mean of the Difference of Random Variables

For any two random variables $X$ and $Y$, if $D = X - Y$, then the expected value of $D$ is

$$E(D) = \mu_D = \mu_X - \mu_Y$$

In general, the mean of the difference of several random variables is the difference of their means. *The order of subtraction is important!*
Subtracting Random Variables: Variance

Variance of the Difference of Random Variables

For any two independent random variables $X$ and $Y$, if $D = X - Y$, then the variance of $D$ is

$$\sigma_D^2 = \sigma_X^2 + \sigma_Y^2$$

In general, the variance of the difference of two independent random variables is the sum of their variances.

**This was an FRQ on the 2013 exam**
Combining Normal Random Variables:

Calculating Probabilities

If a random variable is Normally distributed, we can use its mean and standard deviation to compute probabilities.

**Important Fact:** *Any sum or difference of independent Normal random variables is also Normally distributed.*
Combining Normal Random Variables: Calculating Probabilities

Mrs. Daniel likes between 8.5 and 9 grams of sugar in her iced coffee. Suppose the amount of sugar in a randomly selected packet follows a Normal distribution with mean 2.17 g and standard deviation 0.08 g. If Mrs. Daniel selects 4 packets at random, what is the probability her iced coffee will taste right?
Combining Normal Random Variables: Calculating Probabilities

**STATE & PLAN:** Let $X$ = the amount of sugar in a randomly selected packet. Then, $T = X_1 + X_2 + X_3 + X_4$. We want to find $P(8.5 \leq T \leq 9)$. 

![Normal Distribution Graph]
Combining Normal Random Variables: Calculating Probabilities

DO:
1. Calculate combined mean

\[ \mu_T = \mu_{X_1} + \mu_{X_2} + \mu_{X_3} + \mu_{X_4} = 2.17 + 2.17 + 2.17 + 2.17 = 8.68 \]

2. Calculate combined variance

\[ \sigma_T^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 + \sigma_{X_4}^2 = (0.08)^2 + (0.08)^2 + (0.08)^2 + (0.08)^2 = 0.0256 \]

3. Calculate combined standard deviation.

\[ \sigma_T = \sqrt{0.0256} = 0.16 \]
Combining Normal Random Variables: Calculating Probabilities

4. Normal Calculations:

\[
\text{normcdf} (8.5, 9, 8.68, 0.16) = 0.8470
\]

**CONCLUDE:**
There is an 84.7% percent chance that Mrs. Daniel’s iced coffee will taste right.
Combining Normal Random Variables: Calculating Probabilities

**YES**, you may use your calculator! Just remember to recalculate the combined mean and standard deviation, before using the calculator!!!!
Combining Normal Random Variables: Calculating Probabilities

The diameter $C$ of a randomly selected large drink cup at a fast-food restaurant follows a Normal distribution with a mean of 3.96 inches and a standard deviation of 0.01 inches. The diameter $L$ of a randomly selected large lid at this restaurant follows a Normal distribution with mean 3.98 inches and standard deviation 0.02 inches.

For a lid to fit on a cup, the value of $L$ has to be bigger than the value of $C$, but not by more than 0.06 inches.

What’s the probability that a randomly selected large lid will fit on a randomly chosen large drink cup?
Combining Normal Random Variables: Calculating Probabilities

**STATE & PLAN:** We’ll define the random variable \( D = L - C \) to represent the difference between the lid’s diameter and the cup’s diameter. Our goal is to find \( P(0.00 < D \leq 0.06) \).

**DO:**
1. Calculate combined mean.
   \[
   \mu_D = \mu_L - \mu_C = 3.98 - 3.96 = 0.02
   \]
2. Calculate combined variance
   \[
   (0.02)^2 + (0.01)^2 = 0.0005
   \]
3. Calculate combined standard deviation.
   \[
   \sqrt{0.0005} = 0.0224
   \]
Combining Normal Random Variables: Calculating Probabilities

DO, cont.:

4. normcdf(0, 0.06, 0.02, 0.0224) = 0.7776

CONCLUDE:

We predict that the lids will fit properly 77.76% of the time. This means the lids will not fit properly more than 22% of the time. That is annoying!
Mrs. Daniel and Mrs. Cooper bowl every Tuesday night. Over the past few years, Mrs. Daniel’s scores have been approximately Normally distributed with a mean of 212 and a standard deviation of 31. During the same period, Mrs. Cooper’s scores have also been approximately Normally distributed with a mean of 230 and a standard deviation of 40. Assuming their scores are independent, what is the probability that Mrs. Daniel scores higher than Mrs. Cooper on a randomly-selected Tuesday night?
Combining Normal Random Variables: Calculating Probabilities

**STATE & PLAN:** We’ll define the random variable D as Mrs. Cooper – Mrs. Daniel.

**DO:**
1. Calculate combined mean.
   \[ \mu_D = 230 - 212 = 18 \]
2. Calculate combined variance
   \[ (31)^2 + (40)^2 = 2561 \]
3. Calculate combined standard deviation.
   \[ \sqrt{2561} = 50.606 \]
Combining Normal Random Variables: Calculating Probabilities

DO, cont.
4. Normcdf (-∞, 0.0000001, 18, 50.606) = 0.3610

CONCLUDE:
There is a 36.10% chance that Mrs. Daniel will score higher than Mrs. Cooper on any given night.
Leona and Fred are friendly competitors in high school. Both are about to take the ACT college entrance examination. They agree that if one of them scores 5 or more points better than the other, the loser will buy the winner a pizza. Suppose that in fact Fred and Leona have equal ability, so that each score varies Normally with mean 24 and standard deviation 2. (The variation is due to luck in guessing and the accident of the specific questions being familiar to the student.) The two scores are independent. What is the probability that the scores differ by 5 or more points in either direction?
ACT Scores

New mean: 0
New standard deviation: 2.8284
Normal cdf \((-\infty, -5, 0, 2.8284)\) = 0.0385
+ normcdf\((5, \infty, 0, 2.8284)\) = 0.0385
= 0.0385 + 0.0385 = 0.0771
Mr. Daniel is traveling for his business. He has a new 0.85-ounce tube of toothpaste that’s supposed to last him the whole trip. The amount of toothpaste Mr. Daniel squeezes out of the tube each time he brushes varies according to a Normal distribution with mean 0.13 ounces and standard deviation 0.02 ounces. If Mr. Daniel brushes his teeth six times during the trip, what’s the probability that he’ll be cranky because he ran out of toothpaste?
Toothpaste

New mean: 0.78
New standard deviation: 0.049

\[
\text{Normcdf (0.850000000001, } \infty, 0.78, 0.049) = 0.076564
\]